Bayesian statistics with R
3. Analyses by hand

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Back to Bayes

A simple example

- Let us take a simple example to fix ideas.
- 120 deer were radio-tracked over winter.
- 61 close to a plant, 59 far from any human activity.
- Question: is there a treatment effect on survival?

	Released	Alive	Dead	Other
treatment	61	19	38	4
control	59	21	38	0

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- Of interest is the probability of over-winter survival, call it θ, for the general population within the treatment area.
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- How would the classical statistician justify this estimate?

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- K the number of alive individuals at the end of the winter, so that $P(K = k) = \binom{n}{k}\theta^k(1-\theta)^{n-k}$.
- The classical approach is to maximise the corresponding likelihood with respect to θ to obtain the entirely plausible MLE:

$$\hat{\theta} = k/n = 19/57$$

.

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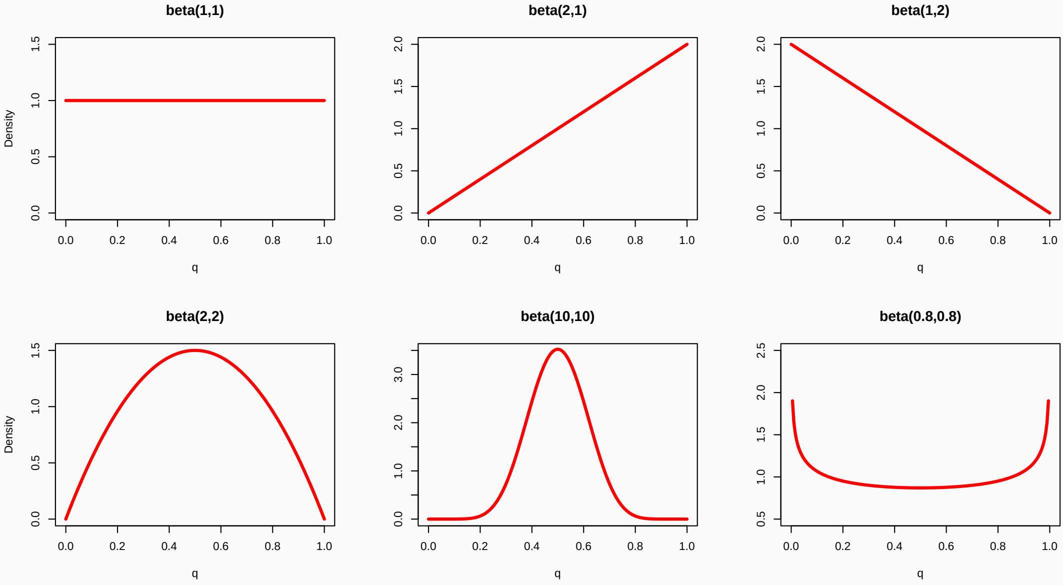
$$q(\theta \mid \alpha, \beta) = \frac{1}{\text{Beta}(\alpha, \beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

with $\text{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and $\Gamma(n) = (n-1)!$


beta(1,1)
beta(2,1)
beta(1,2)
beta(2,2)
beta(10,10)
beta(0.8,0.8)
Density
q

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- Take a Beta prior with a Binomial likelihood, you get a Beta posterior (conjugacy)

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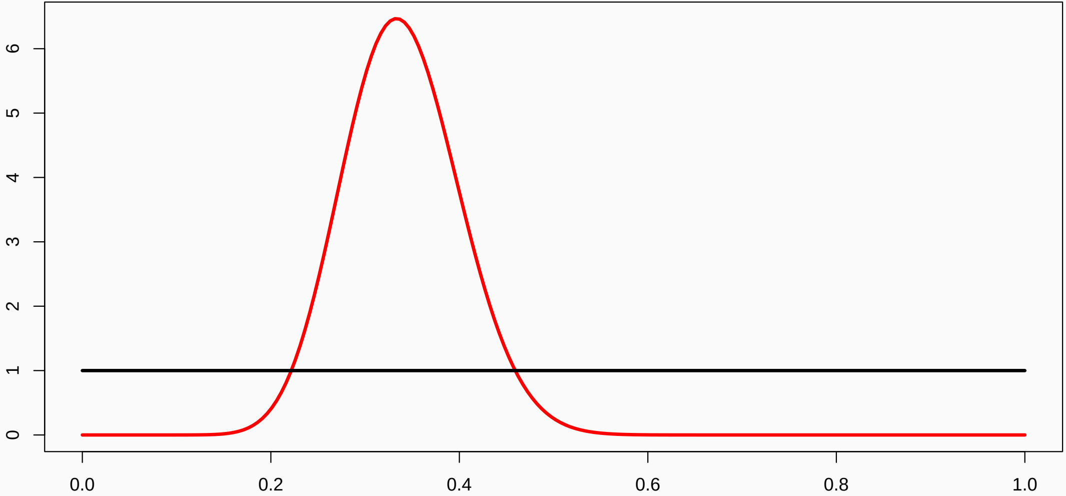
- Note that in this specific situation, the posterior has an explicit expression, easy to manipulate.
- In particular, $E(Beta(a,b)) = \dfrac{a}{a+b} = 20/59$ to be compared with the MLE $19/57$.

A general result

This is a general result, the Bayesian and frequentist estimates will always agree if there is sufficient data, so long as the likelihood is not explicitly ruled out by the prior.

Prior $Beta(1, 1)$ and posterior survival $Beta(20, 39)$



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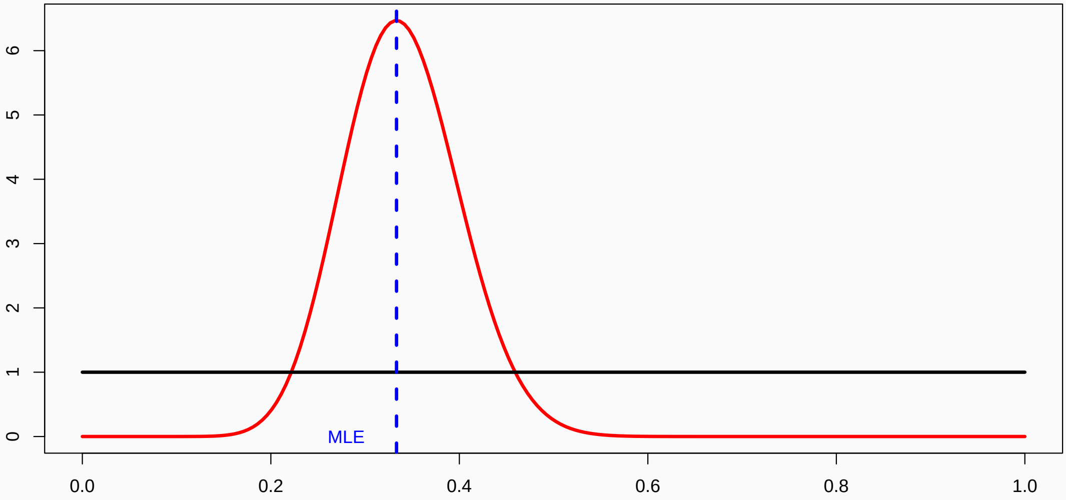
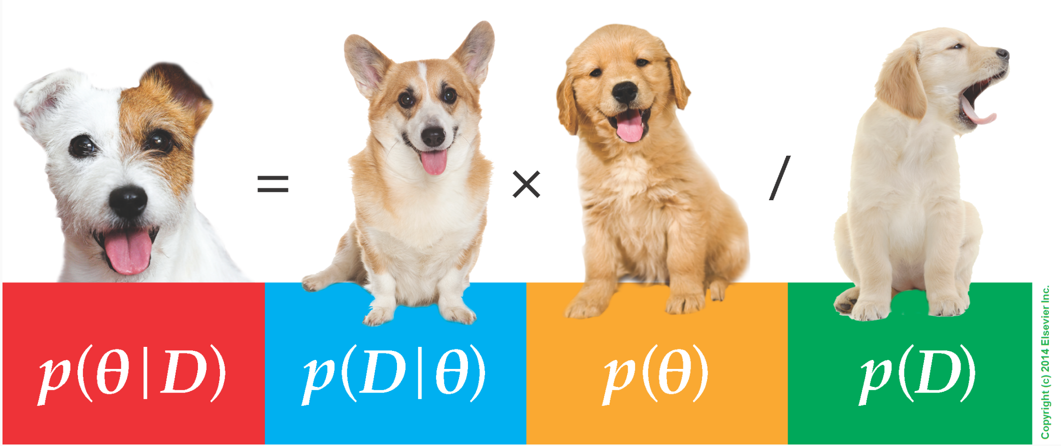


Notation

Our model so far

$$\begin{aligned} y &\sim \text{Binomial}(N, \theta) && \text{[likelihood]} \\ \theta &\sim \text{Beta}(1, 1) && \text{[prior for } \theta\text{]} \end{aligned}$$


$p(\theta|D)$ = $p(D|\theta)$ × $p(\theta)$ / $p(D)$
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