

# The strong reflecting property and Harrington's Principle

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In this paper we characterize the strong reflecting property for  $\mathbf{L}$ -cardinals for all  $\omega_n$ , characterize Harrington's Principle  $\text{HP}(\mathbf{L})$  and its generalization and discuss the relationship between the strong reflecting property for  $\mathbf{L}$ -cardinals and Harrington's Principle  $\text{HP}(\mathbf{L})$ .

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## 1 Introduction and preliminaries

The notion of the strong reflecting property for  $\mathbf{L}$ -cardinals is introduced in [1, Definition 2.5]. The motivation of introducing this notion is to force a set model of Harrington's Principle,  $\text{HP}(\mathbf{L})$  for short (cf. Definition 3.1), over higher order arithmetic (cf. Definition 1.1). However the proof of [1, Main Theorem 1.3] uses very little knowledge about the strong reflecting property for  $\mathbf{L}$ -cardinals. In this paper, in § 2, we develop the full theory of the strong reflecting property for  $\mathbf{L}$ -cardinals and characterize  $\text{SRP}^{\mathbf{L}}(\omega_n)$  for  $n \in \omega$  (cf. Propositions 2.9, & 2.12 and Theorems 2.17 & Theorem 2.23). We also generalize some results on  $\text{SRP}^{\mathbf{L}}(\gamma)$  to  $\text{SRP}^M(\gamma)$  for other inner models  $M$  (cf. Theorems 2.20 & 2.27).

In § 3, we generalize Harrington's Principle to inner models  $M$  and define  $\text{HP}(M)$ ; we give characterizations of  $\text{HP}(M)$  for some well known inner models (cf. Theorems 3.3 & 3.9) and show that, in some cases, this generalized principle fails (cf. Corollary 3.11 and Theorem 3.14). In § 4, we discuss the relationship between the strong reflecting property for  $\mathbf{L}$ -cardinals and Harrington's Principle  $\text{HP}(\mathbf{L})$ .

Our definitions and notations are standard. We refer to textbooks such as [8, 10, 11] for the definitions and notations we use. We write  $\text{tr}(X)$  for the transitive closure of  $X$ . For the definition of admissible set and admissible ordinal, cf. [4]. For notions of large cardinals, cf. [10]. Our notation for forcing is standard (cf. [3, 8]). For the theory of  $0^\#$ , cf. [4, 8]. Recall that  $0^\#$  is the unique well founded remarkable Ehrenfeucht-Mostowski set, and  $0^\#$  exists if and only if for some uncountable limit ordinal  $\lambda$ ,  $\mathbf{L}_\lambda$  has an uncountable set of indiscernibles (cf. [4, 8]). For the theory of  $0^\dagger$ , cf. [10].

**Definition 1.1** (Cheng, [1]) We write  $\text{ZFC}^-$  for ZFC with the Power Set Axiom deleted and Collection instead of Replacement<sup>1</sup> and write  $\text{Z}_2$  for  $\text{ZFC}^-$  + “every set is countable”,  $\text{Z}_3$  for  $\text{ZFC}^-$  + “ $\wp(\omega)$  exists” + “every set is of cardinality  $\leq \beth_1$ ”, and  $\text{Z}_4$  for  $\text{ZFC}^-$  + “ $\wp(\wp(\omega))$  exists” + “every set is of cardinality  $\leq \beth_2$ ”. The axiom systems  $\text{Z}_2$ ,  $\text{Z}_3$  and  $\text{Z}_4$  are the axiomatic systems for second, third, and fourth order arithmetic, respectively.

Throughout this paper whenever we write  $X \prec \mathbf{H}_\kappa$  and  $\gamma \in X$ ,  $\bar{\gamma}$  always denotes the image of  $\gamma$  under the transitive collapse of  $X$ . If  $U$  is an ultrafilter on  $\kappa$ , we say that  $U$  is countably complete if and only if whenever  $Y \subseteq U$  is countable, we have that  $\bigcap Y \neq \emptyset$ . The distinction between  $V$ -cardinals and  $\mathbf{L}$ -cardinals is present throughout the article. Whenever we write  $\omega_n$  (for some  $n$ ) without a superscript it is understood that we mean the  $\omega_n$  of  $V$ . In this paper, a  $\kappa$ -model is a model of the form  $\mathbf{L}[U]$  such that  $(\mathbf{L}[U], \in, U) \models U$  is a normal ultrafilter over  $\kappa$ .

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<sup>1</sup> Cf. [6] for a discussion about the proper axiomatic framework for set theory without the power set axiom.

## 2 Characterizations of the strong reflecting property for $\mathbf{L}$ -cardinals

In this section, we develop the full theory of the strong reflecting property for  $\mathbf{L}$ -cardinals and characterize  $\text{SRP}^{\mathbf{L}}(\omega_n)$  for  $n \in \omega$ . We also generalize some results on  $\text{SRP}^{\mathbf{L}}(\gamma)$  to  $\text{SRP}^M(\gamma)$  for any inner model  $M$ .

Recall that an inner model  $M$  is  $\mathbf{L}$ -like if  $M$  is of the form  $\langle \mathbf{L}[\vec{E}], \in, \vec{E} \rangle$  where  $\vec{E}$  is a coherent sequence of extenders; moreover, for an  $\mathbf{L}$ -like inner model  $M$ ,  $M|\vartheta$  is of the form  $\langle \mathbf{J}_{\vartheta}^{\vec{E}}, \in, \vec{E} \upharpoonright \vartheta, \emptyset \rangle$ .<sup>2</sup>

**Convention 2.1** Throughout, whenever we consider an inner model  $M$  we assume that  $M$  is  $\mathbf{L}$ -like and has the property that  $M|\vartheta$  is definable in  $\mathbf{H}_{\vartheta}$  for any regular cardinal  $\vartheta > \omega_2$ . All known core models satisfy this convention.

**Definition 2.2** Let  $\gamma \geq \omega_1$  be an  $\mathbf{L}$ -cardinal.

- (i) The ordinal  $\gamma$  has the strong reflecting property for  $\mathbf{L}$ -cardinals, denoted  $\text{SRP}^{\mathbf{L}}(\gamma)$ , if and only if for some regular cardinal  $\kappa > \gamma$ , if  $X \prec \mathbf{H}_{\kappa}$ ,  $|X| = \omega$  and  $\gamma \in X$ , then  $\bar{\gamma}$  is an  $\mathbf{L}$ -cardinal.
- (ii) The ordinal  $\gamma$  has the weak reflecting property for  $\mathbf{L}$ -cardinals, denoted  $\text{WRP}^{\mathbf{L}}(\gamma)$ , if and only if for some regular cardinal  $\kappa > \gamma$ , there is  $X \prec \mathbf{H}_{\kappa}$  such that  $|X| = \omega$ ,  $\gamma \in X$  and  $\bar{\gamma}$  is an  $\mathbf{L}$ -cardinal.

**Proposition 2.3** Suppose  $\gamma \geq \omega_1$  is an  $\mathbf{L}$ -cardinal. If  $X \subseteq \gamma$  and  $F : \gamma^{<\omega} \rightarrow \gamma$ , we say that  $X$  is closed under  $F$  if  $F''X^{<\omega} \subseteq X$ . Then the following are equivalent:

- (1)  $\text{SRP}^{\mathbf{L}}(\gamma)$ .
- (2) For any regular cardinal  $\kappa > \gamma$ , if  $X \prec \mathbf{H}_{\kappa}$ ,  $|X| = \omega$  and  $\gamma \in X$ , then  $\bar{\gamma}$  is an  $\mathbf{L}$ -cardinal.
- (3) For some regular cardinal  $\kappa > \gamma$ ,  $\{X \mid X \prec \mathbf{H}_{\kappa}, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } \mathbf{L}\text{-cardinal}\}$  contains a club.
- (4) There exists  $F : \gamma^{<\omega} \rightarrow \gamma$  such that if  $X \subseteq \gamma$  is countable and closed under  $F$ , then  $\text{o.t.}(X)$  is an  $\mathbf{L}$ -cardinal.
- (5) For any regular cardinal  $\kappa > \gamma$ ,  $\{X \mid X \prec \mathbf{H}_{\kappa}, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } \mathbf{L}\text{-cardinal}\}$  contains a club.

**Proof.** Note that (2)  $\Rightarrow$  (1), (1)  $\Rightarrow$  (3), (2)  $\Rightarrow$  (5) and (5)  $\Rightarrow$  (3). It suffices to show that (4)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (4). For the proof, cf. [1, Proposition 2.4].  $\square$

Suppose  $\gamma \geq \omega_1$  is an  $\mathbf{L}$ -cardinal. Let (1)\*, (2)\*, (3)\*, (4)\* and (5)\* be the statements which replace “is an  $\mathbf{L}$ -cardinal” with “is not an  $\mathbf{L}$ -cardinal” in Definition 2.2(i) and statements (2), (3), (4) and (5) in Proposition 2.3, respectively. The following corollary is an observation from the proof of Proposition 2.3.

**Corollary 2.4** The statements (1)\*, (2)\*, (3)\*, (4)\*, and (5)\* are equivalent.

**Proposition 2.5** Suppose  $\gamma \geq \omega_1$  is an  $\mathbf{L}$ -cardinal,  $\kappa$  is regular and  $|\gamma| = \kappa$ . Then the following are equivalent:

- (a)  $\text{SRP}^{\mathbf{L}}(\gamma)$ .
- (b) For any bijection  $\pi : \kappa \rightarrow \gamma$ , there exists a club  $D \subseteq \kappa$  such that for any  $\vartheta \in D$ ,  $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \vartheta\})$  is an  $\mathbf{L}$ -cardinal.
- (c) For some bijection  $\pi : \kappa \rightarrow \gamma$ , there exists a club  $D \subseteq \kappa$  such that for any  $\vartheta \in D$ ,  $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \vartheta\})$  is an  $\mathbf{L}$ -cardinal.

**Proof.** The proof is essentially the same as the case  $\kappa = \omega_1$  in [1, Proposition 2.6].  $\square$

Let (6)\* and (7)\* be the statement which replaces “is an  $\mathbf{L}$ -cardinal” with “is not an  $\mathbf{L}$ -cardinal” in Proposition 2.5(b) and Proposition 2.5(c), respectively. The following corollary is an observation from the proof of Proposition 2.5.

**Corollary 2.6** Suppose  $\gamma \geq \omega_1$  is an  $\mathbf{L}$ -cardinal,  $\kappa$  is regular and  $|\gamma| = \kappa$ . Then both (6)\* and (7)\* are equivalent to (1)\*.

<sup>2</sup> For the definition of coherent sequences of extenders  $\vec{E}$ ,  $\mathbf{J}_{\alpha}^{\vec{E}}$  and  $\vec{E} \upharpoonright \alpha$ , cf. [16, § 2.2].

**Proposition 2.7** *Suppose  $\gamma \geq \omega_1$  is an  $\mathbf{L}$ -cardinal. Then the following are equivalent:*

- (a)  $\text{WRP}^{\mathbf{L}}(\gamma)$ .
- (b) For any regular cardinal  $\kappa > \gamma$ , there is  $X \prec \mathbf{H}_\kappa$  such that  $|X| = \omega$ ,  $\gamma \in X$  and  $\bar{\gamma}$  is an  $\mathbf{L}$ -cardinal.
- (c) For some regular cardinal  $\kappa > \gamma$ ,  $\{X \mid X \prec \mathbf{H}_\kappa, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } \mathbf{L}\text{-cardinal}\}$  is stationary.
- (d) For any  $F : \gamma^{<\omega} \rightarrow \gamma$ , there exists  $X \subseteq \gamma$  such that  $X$  is countable, closed under  $F$  and  $\text{o.t.}(X)$  is an  $\mathbf{L}$ -cardinal.
- (e) For any regular cardinal  $\kappa > \gamma$ ,  $\{X \mid X \prec \mathbf{H}_\kappa, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } \mathbf{L}\text{-cardinal}\}$  is stationary.

**Proof.** Note that (e)  $\Rightarrow$  (c) and (c)  $\Rightarrow$  (a). It suffices to show that (a)  $\Rightarrow$  (d), (d)  $\Rightarrow$  (b), and (b)  $\Rightarrow$  (e). “(a)  $\Rightarrow$  (d)” follows from (4)\*  $\Leftrightarrow$  (2)\* in Corollary 2.4. “(d)  $\Rightarrow$  (b)” follows from (1)\*  $\Leftrightarrow$  (4)\* in Corollary 2.4. “(b)  $\Rightarrow$  (e)” follows from (3)\*  $\Leftrightarrow$  (1)\* in Corollary 2.4.  $\square$

**Proposition 2.8** *Suppose  $\gamma \geq \omega_1$  is an  $\mathbf{L}$ -cardinal,  $\kappa$  is regular and  $|\gamma| = \kappa$ . Then the following are equivalent:*

- (1)  $\text{WRP}^{\mathbf{L}}(\gamma)$ .
- (2) For some bijection  $\pi : \kappa \rightarrow \gamma$ , there exists a stationary  $D \subseteq \kappa$  such that for any  $\vartheta \in D$ ,  $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \vartheta\})$  is an  $\mathbf{L}$ -cardinal.
- (3) For any bijection  $\pi : \kappa \rightarrow \gamma$ , there exists a stationary  $D \subseteq \kappa$  such that for any  $\vartheta \in D$ ,  $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \vartheta\})$  is an  $\mathbf{L}$ -cardinal.

**Proof.** The claim follows from Corollary 2.6 and (1)\*  $\Leftrightarrow$  (2)\* in Corollary 2.4. The proof is standard and we omit the details.  $\square$

**Proposition 2.9** *The following are equivalent:*

- (1)  $\omega_1$  is a limit cardinal in  $\mathbf{L}$ .
- (2)  $\text{WRP}^{\mathbf{L}}(\omega_1)$ .
- (3)  $\text{SRP}^{\mathbf{L}}(\omega_1)$ .

**Proof.** It suffices to show that (1)  $\Rightarrow$  (3) and (2)  $\Rightarrow$  (1) since (3)  $\Rightarrow$  (2) is immediate.

“(1)  $\Rightarrow$  (3)”. Suppose  $\omega_1$  is a limit cardinal in  $\mathbf{L}$ . Then  $\{\alpha < \omega_1 \mid \alpha \text{ is an } \mathbf{L}\text{-cardinal}\}$  is a club. By Proposition 2.5,  $\text{SRP}^{\mathbf{L}}(\omega_1)$  holds.

“(2)  $\Rightarrow$  (1)”. Suppose  $\text{WRP}^{\mathbf{L}}(\omega_1)$  holds. Then  $\{X \cap \omega_1 \mid X \prec \mathbf{H}_{\omega_2} \text{ and } |X| = \omega \text{ and } \text{o.t.}(X \cap \omega_1) \text{ is an } \mathbf{L}\text{-cardinal}\}$  is stationary in  $\omega_1$ . It is easy to see that for any  $\alpha < \omega_1$  there is  $\alpha < \beta < \omega_1$  such that  $\beta$  is an  $\mathbf{L}$ -cardinal.  $\square$

**Proposition 2.10** *Suppose  $\gamma \geq \omega_1$  is an  $\mathbf{L}$ -cardinal,  $\kappa > \gamma$  is a regular cardinal and  $\text{SRP}^{\mathbf{L}}(\gamma)$  holds. If  $Z \prec \mathbf{H}_\kappa$ ,  $|Z| \leq \omega_1$  and  $\gamma \in Z$ , then  $\bar{\gamma}$  is an  $\mathbf{L}$ -cardinal.*

**Proof.** Suppose  $\bar{\gamma}$  is not an  $\mathbf{L}$ -cardinal. Let  $M$  be the transitive collapse of  $Z$  and  $\pi : M \prec \mathbf{H}_\kappa$  be the inverse of the collapsing map. Take  $Y \prec \mathbf{H}_\kappa$  such that  $|Y| = \omega$  and  $M, \bar{\gamma} \in Y$ . Note that  $Y \models \text{“}\bar{\gamma} \text{ is not an } \mathbf{L}\text{-cardinal”}$ . If  $\bar{\bar{\gamma}}$  is the image of  $\bar{\gamma}$  under the transitive collapse of  $Y$ , then  $\bar{\bar{\gamma}}$  is not an  $\mathbf{L}$ -cardinal. Let  $X = \pi''(Y \cap M)$ . Since  $\bar{\gamma} \in Y \cap M$  and  $\pi(\bar{\gamma}) = \gamma$ , we have  $\gamma \in X$ . Note that  $X \prec Z \prec \mathbf{H}_\kappa$  and the image of  $\gamma$  under the transitive collapse of  $X$  is  $\bar{\bar{\gamma}}$ . By  $\text{SRP}^{\mathbf{L}}(\gamma)$ ,  $\bar{\bar{\gamma}}$  is an  $\mathbf{L}$ -cardinal. Contradiction.  $\square$

**Proposition 2.11** *Suppose  $\omega_1 \leq \gamma_0 < \gamma_1$  are  $\mathbf{L}$ -cardinals. Then  $\text{SRP}^{\mathbf{L}}(\gamma_1)$  implies  $\text{SRP}^{\mathbf{L}}(\gamma_0)$  and  $\text{WRP}^{\mathbf{L}}(\gamma_1)$  implies  $\text{WRP}^{\mathbf{L}}(\gamma_0)$ .*

**Proof.** We only show the strong reflecting property case (the argument for the weak reflecting property case is similar). Let  $\kappa > \gamma_1$  be a regular cardinal. It suffices to show if  $X \prec \mathbf{H}_\kappa$ ,  $|X| = \omega$  and  $\{\gamma_0, \gamma_1\} \subseteq X$ , then  $\bar{\gamma}_0$  is an  $\mathbf{L}$ -cardinal. Note that  $\mathbf{L}_{\gamma_1} \models \gamma_0$  is a cardinal. Since  $\gamma_1 \in X$ , we have  $\mathbf{L}_{\gamma_1} \in X$ . Since  $\overline{\mathbf{L}_{\gamma_1}} = \mathbf{L}_{\bar{\gamma}_1}$  and  $\overline{\mathbf{L}_{\gamma_1}} \models \bar{\gamma}_0$  is a cardinal, we have that  $\mathbf{L}_{\bar{\gamma}_1} \models \bar{\gamma}_0$  is a cardinal. By  $\text{SRP}^{\mathbf{L}}(\gamma_1)$ ,  $\bar{\gamma}_1$  is an  $\mathbf{L}$ -cardinal and hence  $\bar{\gamma}_0$  is an  $\mathbf{L}$ -cardinal.  $\square$

**Proposition 2.12** *The following are equivalent:*

- (1)  $\text{SRP}^{\mathbf{L}}(\omega_2)$ .

- (2)  $\omega_2$  is a limit cardinal in  $\mathbf{L}$  and for any  $\mathbf{L}$ -cardinal  $\omega_1 \leq \gamma < \omega_2$ ,  $\text{SRP}^{\mathbf{L}}(\gamma)$  holds.
- (3)  $\{\alpha < \omega_2 \mid \alpha \text{ is an } \mathbf{L}\text{-cardinal and } \text{SRP}^{\mathbf{L}}(\alpha) \text{ holds}\}$  is unbounded in  $\omega_2$ .

**Proof.** “(1)  $\Rightarrow$  (2)”. By Proposition 2.11, it suffices to show  $\omega_2$  is a limit cardinal in  $\mathbf{L}$ . Let  $\kappa > \omega_2$  be the regular cardinal that witnesses  $\text{SRP}^{\mathbf{L}}(\omega_2)$ . Fix  $\alpha < \omega_2$ . Pick  $Z \prec \mathbf{H}_\kappa$  such that  $|Z| = \omega_1$ ,  $\alpha \subseteq Z$  and  $\omega_2 \in Z$ . By Proposition 2.10,  $\bar{\omega}_2$  is an  $\mathbf{L}$ -cardinal. Note that  $\alpha \leq \bar{\omega}_2 < \omega_2$ .

“(2)  $\Rightarrow$  (1)”. Suppose  $\kappa > \omega_2$  is a regular cardinal,  $X \prec \mathbf{H}_\kappa$ ,  $|X| = \omega$  and  $\omega_2 \in X$ . We show that  $\bar{\omega}_2$  is an  $\mathbf{L}$ -cardinal. Note that  $\bar{\omega}_2 = \text{o.t.}(X \cap \omega_2)$ . Let  $E = \{\gamma \mid \omega_1 \leq \gamma < \omega_2 \wedge \gamma \text{ is an } \mathbf{L}\text{-cardinal}\}$ . The set  $E$  is definable in  $\mathbf{H}_\kappa$ . Since  $\omega_2$  is a limit cardinal in  $\mathbf{L}$ ,  $E$  is cofinal in  $\omega_2$  and hence  $E \cap X$  is cofinal in  $\omega_2 \cap X$ . For  $\gamma \in E \cap X$ ,  $\bar{\gamma} = \text{o.t.}(X \cap \gamma)$  and by  $\text{SRP}^{\mathbf{L}}(\gamma)$ ,  $\bar{\gamma}$  is an  $\mathbf{L}$ -cardinal. Note that  $\bar{\omega}_2 = \sup(\{\bar{\gamma} \mid \gamma \in E \cap X\})$ . Hence  $\bar{\omega}_2$  is an  $\mathbf{L}$ -cardinal.

“(1)  $\Leftrightarrow$  (3)” follows from (1)  $\Leftrightarrow$  (2) and Proposition 2.11. □

The notion of remarkable cardinal was introduced by Schindler in [15]. Any remarkable cardinal is remarkable in  $\mathbf{L}$  (cf. [15, Lemma 1.7]):

Let  $\kappa$  be a cardinal,  $G$  be  $\text{Col}(\omega, < \kappa)$ -generic over  $V$ ,  $\vartheta > \kappa$  be a regular cardinal and  $X \in [\mathbf{H}_\vartheta^{V[G]}]^\omega$ . We say that  $X$  condenses remarkably if  $X = \text{ran}(\pi)$  for some elementary  $\pi : (\mathbf{H}_\beta^{V[G \cap \mathbf{H}_\alpha^V]}, \in, \mathbf{H}_\beta^V, G \cap \mathbf{H}_\alpha^V) \rightarrow (\mathbf{H}_\vartheta^{V[G]}, \in, \mathbf{H}_\vartheta^V, G)$  where  $\alpha = \text{crit}(\pi) < \beta < \kappa$  and  $\beta$  is a cardinal in  $V$ .

For regular cardinal  $\vartheta > \kappa$ ,  $\kappa$  is  $\vartheta$ -remarkable if and only if in  $V^{\text{Col}(\omega, < \kappa)}$ ,  $\{X \in [\mathbf{H}_\vartheta]^\omega : X \text{ condenses remarkably}\}$  is stationary. We say that  $\kappa$  is remarkable if  $\kappa$  is  $\vartheta$ -remarkable for all regular cardinal  $\vartheta > \kappa$ .

**Lemma 2.13** (Cheng, [1, Lemma 2.2]) *Suppose  $\kappa$  is an  $\mathbf{L}$ -cardinal. The following are equivalent:*

- (1)  $\kappa$  is remarkable in  $\mathbf{L}$ ;
- (2) If  $\gamma \geq \kappa$  is an  $\mathbf{L}$ -cardinal,  $\vartheta > \gamma$  is a regular cardinal in  $\mathbf{L}$ , then  $\Vdash_{\text{Col}(\omega, < \kappa)}^{\mathbf{L}} \text{“}\{X \mid X \prec \mathbf{L}_\vartheta^{\dot{G}}, |X| = \omega \text{ and } \text{o.t.}(X \cap \check{\gamma}) \text{ is an } \mathbf{L}\text{-cardinal}\} \text{ is stationary”}$ .

**Corollary 2.14** *If  $\kappa$  is remarkable in  $\mathbf{L}$  and  $G$  is  $\text{Col}(\omega, < \kappa)$ -generic over  $\mathbf{L}$ , then  $\mathbf{L}[G] \models \text{WRP}^{\mathbf{L}}(\gamma)$  holds for any  $\mathbf{L}$ -cardinal  $\gamma \geq \kappa$ .*

**Proof.** The claim follows from Lemma 2.13. □

Fix some  $\mathbf{L}$ -cardinal  $\gamma \geq \omega_1$ . The statement  $\text{SRP}^{\mathbf{L}}(\gamma)$  is upward absolute (cf. [1, Proposition 2.8]; the key point is that Proposition 2.3(4) is upward absolute). As a corollary,  $\text{WRP}^{\mathbf{L}}(\gamma)$  is downward absolute (the key point is that Proposition 2.7(d) is downward absolute). So if  $\text{WRP}^{\mathbf{L}}(\gamma)$  holds, then  $\text{WRP}^{\mathbf{L}}(\gamma)$  holds in  $\mathbf{L}$ . The converse is not true in general.

**Proposition 2.15** *Suppose  $\text{WRP}^{\mathbf{L}}(\kappa)$  holds where  $\kappa \geq \omega_1$  is an  $\mathbf{L}$ -cardinal. Then  $\mathbf{L} \models \text{“}\omega_1 \text{ is } \kappa^+\text{-remarkable”}$  and for any regular  $\vartheta > \kappa$  in  $\mathbf{L}$ , we have that  $\mathbf{L} \models \text{“}\omega_1 \text{ is } \vartheta\text{-remarkable”}$ .*

**Proof.** We have that  $\mathbf{L} \models \text{WRP}^{\mathbf{L}}(\kappa)$  iff  $\{X \mid X \prec \mathbf{L}_{\kappa^+}, |X| = \omega \text{ and } \text{o.t.}(X \cap \kappa) \text{ is an } \mathbf{L}\text{-cardinal}\}$  is stationary in  $\mathbf{L}$  iff for any  $\mathbf{L}$ -regular cardinal  $\vartheta > \kappa$ ,  $\{X \mid X \prec \mathbf{L}_\vartheta, |X| = \omega \text{ and } \text{o.t.}(X \cap \kappa) \text{ is an } \mathbf{L}\text{-cardinal}\}$  is stationary in  $\mathbf{L}$ . For  $\mathbf{L}$ -regular cardinal  $\vartheta > \kappa$ , we have that  $\mathbf{L} \models \text{“}\omega_1 \text{ is } \vartheta\text{-remarkable”}$  iff for any  $G$  which is  $\text{Col}(\omega, < \omega_1)$ -generic over  $\mathbf{L}$ ,  $\mathbf{L}[G] \models \{X \in [\mathbf{L}_\vartheta]^\omega \mid X = \text{ran}(\pi), \pi : (\mathbf{L}_\beta[G \upharpoonright \alpha], \in, \mathbf{L}_\beta, G \upharpoonright \alpha) \prec (\mathbf{L}_\vartheta[G], \in, \mathbf{L}_\vartheta, G) \text{ where } \alpha = \text{crit}(\pi) < \beta < \omega_1 \text{ and } \beta \text{ is an } \mathbf{L}\text{-cardinal}\}$  is stationary. Note that  $\mathbf{L} \models \text{WRP}^{\mathbf{L}}(\kappa)$  and  $\text{Col}(\omega, < \omega_1)$  is stationary preserving. □

**Corollary 2.16** *“For any  $\mathbf{L}$ -cardinal  $\gamma \geq \omega_1$ ,  $\text{WRP}^{\mathbf{L}}(\gamma)$  holds” is equiconsistent with “ $\omega_1$  is remarkable”.*

**Proof.** The claim follows from Corollary 2.14 and Proposition 2.15. □

**Theorem 2.17** (Set forcing) *The following two theories are equiconsistent:*

- (1)  $\text{SRP}^{\mathbf{L}}(\omega_2)$ .
- (2)  $\text{ZFC} + \text{“there exists a remarkable cardinal with a weakly inaccessible cardinal above it”}$ .

**Proof.** We first show that the consistency of (2) implies the consistency of (1). Let  $S = \{\omega_1 \leq \alpha < \omega_2 \mid \alpha \text{ is an } \mathbf{L}\text{-cardinal}\}$ . Note that  $\text{SRP}^{\mathbf{L}}(\omega_2)$  is equivalent to  $S$  being a club such that  $\text{SRP}^{\mathbf{L}}(\alpha)$  holds for any  $\alpha \in S$ . In [1, § 3.1], assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, we force a model  $\mathbf{L}[G, H]$  in which  $S$  is a club and  $\text{SRP}^{\mathbf{L}}(\alpha)$  holds for any  $\alpha \in S$ . So  $\text{SRP}^{\mathbf{L}}(\omega_2)$  holds in  $\mathbf{L}[G, H]$ .

From [1, §§ 3.2–3.4], if  $S$  is a club and  $\text{SRP}^{\mathbf{L}}(\alpha)$  holds for any  $\alpha \in S$ , then we can force a model of  $Z_3 + \text{HP}(\mathbf{L})$ . So the consistency of (1) implies the consistency of  $Z_3 + \text{HP}(\mathbf{L})$ . By [2, Theorem 3.2],  $Z_3 + \text{HP}(\mathbf{L})$  implies that  $\mathbf{L} \models \text{ZFC} + \text{“}\omega_1^V \text{ is remarkable”}$ . By Proposition 2.12,  $\omega_2^V$  is inaccessible in  $\mathbf{L}$ . So the consistency of (1) implies the consistency of (2).  $\square$

**Definition 2.18** Suppose  $M$  is an inner model and  $\gamma \geq \omega_1$  is an  $M$ -cardinal. We say that  $\gamma$  has the *strong reflecting property for  $M$ -cardinals*, denoted  $\text{SRP}^M(\gamma)$ , if and only if for some regular cardinal  $\kappa > \gamma$ , if  $X < \mathbf{H}_\kappa$ ,  $|X| = \omega$  and  $\gamma \in X$ , then  $\bar{\gamma}$  is an  $M$ -cardinal.

**Definition 2.19** Suppose  $M$  is an inner model. We say that  $M$  has the *full covering property* if for any set  $X$  of ordinals, there is  $Y \in M$  such that  $X \subseteq Y$  and  $|Y| = |X| + \omega_1$ . We say that  $M$  has the *rigidity property* if there is no nontrivial elementary embedding from  $M$  to  $M$ .

**Theorem 2.20** Suppose  $M$  is an inner model which satisfies Convention 2.1 and has both the full covering and the rigidity property. Then, for every  $M$ -cardinal  $\gamma > \omega_2$ ,  $\text{SRP}^M(\gamma)$  fails.

**Proof.** Suppose  $\text{SRP}^M(\gamma)$  holds for some  $\gamma > \omega_2$ . Let  $\kappa > \gamma$  be the witnessing regular cardinal for  $\text{SRP}^M(\gamma)$ . Build an elementary chain  $\langle Z_\alpha \mid \alpha < \omega_1 \rangle$  of submodels of  $\mathbf{H}_\kappa$  such that for all  $\alpha < \beta < \omega_1$ ,  $Z_\alpha < Z_\beta < \mathbf{H}_\kappa$ ,  $Z_\alpha \in Z_\beta$ ,  $|Z_\alpha| = \omega$  and  $\{\gamma, \omega_2\} \subseteq Z_0$ .

Let  $Z = \bigcup_{\alpha < \omega_1} Z_\alpha$ . Then  $|Z| = \omega_1$  and  $Z < \mathbf{H}_\kappa$ . Let  $\pi : N \cong Z < \mathbf{H}_\kappa$  and  $\pi_\alpha : N_\alpha \cong Z_\alpha < \mathbf{H}_\kappa$  be the inverses of the collapsing maps. Let  $j_\alpha : N_\alpha \rightarrow N$  be the induced elementary embedding. Since  $\omega_1 \subseteq Z$ , we have that  $\text{crit}(\pi) > \bar{\omega}_1$ . Since  $\omega_2 \in Z$  and  $|Z| = \omega_1$ , we have that  $\text{crit}(\pi) \leq \bar{\omega}_2$ . So  $\text{crit}(\pi) = \bar{\omega}_2$ .

Note that Proposition 2.10 still holds if we replace  $\mathbf{L}$  with  $M$ . By  $\text{SRP}^M(\gamma)$ ,  $\bar{\gamma}$  is an  $M$ -cardinal. Since  $M \upharpoonright \bar{\gamma}$  is definable in  $\mathbf{H}_\kappa$ , we have that  $\wp(\bar{\omega}_2) \cap M \subseteq M \upharpoonright \bar{\gamma} \in N$  and  $\wp(\bar{\omega}_2) \cap M \in N$ . Define  $U = \{X \subseteq \bar{\omega}_2 \mid X \in M \wedge \bar{\omega}_2 \in \pi(X)\}$ . Then  $U$  is an  $M$ -ultrafilter. For  $\alpha < \omega_1$ , the image of  $Z_\alpha$  under the transitive collapse of  $Z$  is  $j_\alpha''N_\alpha$  and  $j_\alpha''N_\alpha \in N$ .  $\square$

**Lemma 2.21** The  $M$ -ultrafilter  $U$  is countably complete.

**Proof.** Suppose  $Y \subseteq U$  and  $Y$  is countable. We show that  $\bigcap Y \neq \emptyset$ . Since  $Y \subseteq N$ , take  $\alpha < \omega_1$  large enough such that  $Y \subseteq j_\alpha''N_\alpha$ . Let  $S = \wp(\bar{\omega}_2) \cap M \cap j_\alpha''N_\alpha$ . Note that  $S \in N$  and  $N \models S$  is countable.

Using that  $M \upharpoonright \vartheta$  is definable in  $\mathbf{H}_\vartheta$  for regular cardinal  $\vartheta > \omega_2$ , we note that  $\mathbf{H}_\kappa \models \text{“}M \text{ has the full covering property”}$  whence  $N \models M$  has the full covering property. Fix  $T \in N$  such that  $T \subseteq \wp(\bar{\omega}_2) \cap M$ ,  $T \supseteq S$ ,  $T \in M$  and  $N \models |T| = \omega_1$ . Since  $\bar{\omega}_2 = \text{crit}(\pi) > \omega_1$ , we have that  $\pi(T) = \pi''T$ . Since  $T \in N$ , we have that  $\wp(T) \cap M \in N$ .  $\square$

**Claim 2.22** We have that  $U \cap T \in N$ .

**Proof.** Since  $\pi(T) = \pi''T \in M$ , we have that  $\pi''(U \cap T) = \{\pi(A) \mid A \in T \wedge \bar{\omega}_2 \in \pi(A)\} = \{B \in \pi(T) \mid \bar{\omega}_2 \in B\}$  and  $\pi''(U \cap T) \in M$ . Note that  $\wp(\pi''T) \cap M = \pi''(\wp(T) \cap M)$  since for all  $D \in \wp(T) \cap M$ ,  $\pi(D) = \pi''D$ . Since  $\pi''(U \cap T) \in \wp(\pi''T) \cap M$ , we have that  $\pi''(U \cap T) = \pi(D) = \pi''D$  for some  $D \in \wp(T) \cap M \subseteq N$ . So  $U \cap T = D$  and hence  $U \cap T \in N$ .  $\square$

Note that  $Y \subseteq j_\alpha''N_\alpha \cap \wp(\bar{\omega}_2) \cap M = S \subseteq T$ . Since  $Y \subseteq T \cap U$ , to show that  $\bigcap Y \neq \emptyset$ , it suffices to show that  $\bigcap(U \cap T) \neq \emptyset$ . Note that  $\bar{\omega}_2 \in \bigcap \pi''(U \cap T)$  and  $\pi(U \cap T) = \pi''(U \cap T)$ . Then  $\bigcap \pi''(U \cap T) = \bigcap \pi(U \cap T) = \pi(\bigcap(U \cap T)) \neq \emptyset$ . So  $\bigcap(U \cap T) \neq \emptyset$ .  $\square$

So we can build a nontrivial embedding from  $M$  to  $M$  which contradicts the rigidity property of  $M$ .  $\square$

**Theorem 2.23** The following are equivalent:

- (i)  $\text{SRP}^{\mathbf{L}}(\gamma)$  holds for some  $\mathbf{L}$ -cardinal  $\gamma > \omega_2$ .
- (ii)  $0^\#$  exists.
- (iii)  $\text{SRP}^{\mathbf{L}}(\gamma)$  holds for every  $\mathbf{L}$ -cardinal  $\gamma \geq \omega_1$ .

**Proof.** “(i)  $\Rightarrow$  (ii)”. Assume  $0^\#$  does not exist. Then  $\mathbf{L}$  satisfies all the conditions for  $M$  in Theorem 2.20. From the proof of Theorem 2.20 (replace  $M$  with  $\mathbf{L}$ ),  $\text{SRP}^{\mathbf{L}}(\gamma)$  does not hold for any  $\mathbf{L}$  cardinal  $\gamma > \omega_2$ .

“(ii)  $\Rightarrow$  (iii)”. We write  $\mathcal{M}(0^\#, \alpha)$  for the unique transitive  $(0^\#, \alpha)$ -model (cf. [10] for definitions). Note that if  $X < \mathbf{H}_\kappa$  and  $\gamma \in X$ , then  $\mathcal{M}(0^\#, \gamma + 1) \in X$  and its image under the transitive collapse of  $X$  is  $\mathcal{M}(0^\#, \bar{\gamma} + 1)$ . Note that for  $\alpha \in \text{Ord}$ ,  $\mathcal{M}(0^\#, \alpha) < L$ .  $\square$

So for  $n \geq 3$ ,  $\text{SRP}^{\mathbf{L}}(\omega_n)$  is equivalent to “ $0^\#$  exists”. We have characterized  $\text{SRP}^{\mathbf{L}}(\omega_n)$  for  $n \geq 1$ .

**Definition 2.24** Suppose  $M$  is an inner model. For  $M$ -cardinal  $\lambda$ , let  $\text{SRP}_{<\lambda}^M(\lambda)$  denote the statement: “for some regular cardinal  $\vartheta > \lambda$ , if  $X < \mathbf{H}_\vartheta$ ,  $|X| < \lambda$  and  $\lambda \in X$ , then  $\bar{\lambda}$  is an  $M$ -cardinal”.

**Fact 2.25** Assume  $0^\dagger$  does not exist but there is an inner model with a measurable cardinal and  $\mathbf{L}[U]$  is chosen such that  $\kappa = \text{crit}(U)$  is as small as possible. Then one of the following holds:

- (a) For every set  $X$  of ordinals, there is a set  $Y \in \mathbf{L}[U]$  such that  $Y \supseteq X$  and  $|Y| = |X| + \omega_1$ ;
- (b) There is a sequence  $C \subseteq \kappa$ , which is Příkrý generic over  $\mathbf{L}[U]$ , such that for all set  $X$  of ordinals, there is a set  $Y \in \mathbf{L}[U, C]$  such that  $Y \supseteq X$  and  $|Y| = |X| + \omega_1$ .

For a proof, cf. [13, Theorem 1.3].

**Fact 2.26** The following are equivalent:

- (1)  $0^\dagger$  exists.
- (2) There is a  $\kappa$ -model for some  $\kappa$  and an elementary embedding from that model to itself with critical point greater than  $\kappa$ .

Cf. [10, Exercise 21.22].

**Theorem 2.27** Suppose there is an inner model with a measurable cardinal and  $\mathbf{L}[U]$  is chosen such that  $\kappa = \text{crit}(U)$  is as small as possible. Suppose  $\lambda > \kappa^+$  is an  $\mathbf{L}[U]$ -cardinal. Then  $\text{SRP}_{<\lambda}^{\mathbf{L}[U]}(\lambda)$  if and only if  $0^\dagger$  exists.

**Proof.** “ $\Rightarrow$ ”. We assume that  $0^\dagger$  does not exist and try to get a contradiction. By Fact 2.25, we need to discuss two cases.

**Case 1.** Fact 2.25(a) holds. Let  $\vartheta > \lambda$  be the witness regular cardinal for  $\text{SRP}_{<\lambda}^{\mathbf{L}[U]}(\lambda)$ . Build an elementary chain  $\langle Z_\alpha \mid \alpha < \kappa \rangle$  of submodels of  $\mathbf{H}_\vartheta$  such that for  $\alpha < \beta < \kappa$ , we have  $Z_\alpha < Z_\beta < \mathbf{H}_\vartheta$ ,  $Z_\alpha \in Z_\beta$ ,  $|Z_\alpha| = \kappa$ , and  $\{\kappa^+, \lambda\} \cup \text{tr}(\{U\}) \subseteq Z_0$ . Let  $Z = \bigcup_{\alpha < \kappa} Z_\alpha$ . Then  $|Z| = \kappa$ . Let  $\pi : N \cong Z < \mathbf{H}_\vartheta$  and  $\pi_\alpha : N_\alpha \cong Z_\alpha < \mathbf{H}_\vartheta$  be the inverses of the collapsing maps. Since  $Z_\alpha < Z$ , let  $j_\alpha : N_\alpha < N$  be the induced embedding. Then  $\pi_\alpha = \pi \circ j_\alpha$  and  $N = \bigcup_{\alpha < \kappa} j_\alpha''N_\alpha$ . Let  $\text{crit}(\pi) = \eta$ . Then  $\eta > \kappa = \bar{\kappa}$  and since  $|Z| = \kappa$ , we have  $\eta \leq \kappa^+$ . So  $\eta = \kappa^+ < \bar{\lambda}$ . By  $\text{SRP}_{<\lambda}^{\mathbf{L}[U]}(\lambda)$ ,  $\bar{\lambda}$  is an  $\mathbf{L}[U]$ -cardinal. Let  $W = \{X \subseteq \eta \mid X \in \mathbf{L}[U] \text{ and } \eta \in \pi(X)\}$ . Note that  $U = \bar{U} \in N$  and  $W \subseteq \mathbf{L}_{\bar{\lambda}}[U] \subseteq N$ . Then  $W$  is an  $\mathbf{L}[U]$ -ultrafilter on  $\eta$ . Note that  $Z \models “|Z_\alpha| = \kappa”$  and the image of  $Z_\alpha$  under the transitive collapse of  $Z$  is  $j_\alpha''N_\alpha$ . So for  $\alpha < \kappa$ , we have  $j_\alpha''N_\alpha \in N$  and  $N \models “|j_\alpha''N_\alpha| = \kappa”$ .

**Lemma 2.28** The filter  $W$  is countably complete.

**Proof.** Suppose  $Y \subseteq W$  and  $Y$  is countable. We show that  $\bigcap Y \neq \emptyset$ . Since  $Y \subseteq N$ , take  $\alpha < \kappa$  large enough such that  $Y \subseteq j_\alpha''N_\alpha$ . Let  $S = \wp(\eta) \cap \mathbf{L}[U] \cap j_\alpha''N_\alpha$ . Note that  $\wp(\eta) \cap \mathbf{L}[U] \in N$  and hence  $S \in N$ . We have that  $N \models |S| \leq \kappa$ . Since Fact 2.25(a) holds in  $\mathbf{H}_\vartheta$  and  $N < \mathbf{H}_\vartheta$ , Fact 2.25(a) holds in  $N$ . Take  $T \in N$  such that  $T \subseteq \wp(\eta) \cap \mathbf{L}[U]$ ,  $T \supseteq S$ ,  $T \in \mathbf{L}[U]$ , and  $N \models |T| \leq \kappa$ . Since  $\eta > \kappa$ , we have  $\pi(T) = \pi''T$ . Let  $\bar{T} = \{X \in T \mid \eta \in \pi(X)\}$ .

**Claim 2.29** We have that  $\bar{T} \in N$ .

**Proof.** Since  $N \models |T| \leq \kappa$ , there is  $h \in N$  such that  $h : T \leftrightarrow \gamma$  for some  $\gamma < \eta$ . Then  $\bar{T} = \{X \in T \mid \eta \in \pi''(h^{-1})(h(X))\}$ . So  $\bar{T} \in N$ .  $\square$

Note that  $\bigcap \bar{T} \neq \emptyset$  since  $\pi(\bar{T}) = \pi''\bar{T}$  and  $\eta \in \bigcap \pi''\bar{T} = \bigcap \pi(\bar{T}) = \pi(\bigcap \bar{T})$ . Since  $Y \subseteq S \subseteq T$  and  $Y \subseteq W$ , we have  $Y \subseteq \bar{T}$  and hence  $\bigcap Y \neq \emptyset$ .  $\square$

So there exists a nontrivial elementary embedding  $j : \mathbf{L}[U] < \mathbf{L}[U]$  with  $\text{crit}(j) = \eta > \kappa$ . By Fact 2.26,  $0^\dagger$  exists. Contradiction.

**Case 2.** Fact 2.25(b) holds. The proof is essentially the same as *Case 1* with small modifications (e.g., let  $\text{tr}(\{U, C\}) \subseteq Z_0$  and  $W = \{X \subseteq \eta \mid X \in \mathbf{L}[U, C] \text{ and } \eta \in \pi(X)\}$ ). Since Příkrý forcing preserves all cardinals,  $\bar{\lambda}$  is an  $\mathbf{L}[U, C]$ -cardinal. As in Case 1, we can show that there exists a nontrivial elementary embedding  $j : \mathbf{L}[U, C] \prec \mathbf{L}[U, C]$ . Since  $j(U, C) = (U, C)$ , we have  $j \upharpoonright \mathbf{L}[U] : \mathbf{L}[U] \prec \mathbf{L}[U]$  with  $\text{crit}(j \upharpoonright \mathbf{L}[U]) = \eta > \kappa$ . So by Fact 2.26,  $0^\dagger$  exists. Contradiction.

“ $\Leftarrow$ ”. Assume  $0^\dagger$  exists. Suppose  $\vartheta > \lambda$  is regular,  $X \prec \mathbf{H}_\vartheta$ ,  $|X| < \lambda$  and  $\lambda \in X$ . We show that  $\bar{\lambda}$  is an  $\mathbf{L}[U]$ -cardinal. Since  $\lambda \in X$  and  $0^\dagger \in X$ , we have  $\mathcal{M}(0^\dagger, \omega, \lambda + 1) \in X$ .<sup>3</sup> Note that for any  $\alpha, \beta \in \text{Ord}$ ,  $\mathcal{M}(0^\dagger, \alpha, \beta) \prec \mathbf{L}[U]$ . Since  $\lambda$  is an  $\mathbf{L}[U]$ -cardinal and  $\lambda \in \mathcal{M}(0^\dagger, \omega, \lambda + 1)$ , we have  $\mathcal{M}(0^\dagger, \omega, \lambda + 1) \models \text{“}\lambda \text{ is a cardinal”}$ . Note that the image of  $\mathcal{M}(0^\dagger, \omega, \lambda + 1)$  under the transitive collapse of  $X$  is  $\mathcal{M}(0^\dagger, \omega, \bar{\lambda} + 1)$ . So  $\mathcal{M}(0^\dagger, \omega, \bar{\lambda} + 1) \models \text{“}\bar{\lambda} \text{ is a cardinal”}$ . Since  $\mathcal{M}(0^\dagger, \omega, \bar{\lambda} + 1) \prec \mathbf{L}[U]$ , we have that  $\bar{\lambda}$  is an  $\mathbf{L}[U]$ -cardinal.  $\square$

In [14], Räsch and Schindler introduced the condensation principle  $\nabla_\kappa$ : for any regular cardinal  $\vartheta > \kappa$ ,  $\{X \prec \mathbf{L}_\vartheta \mid |X| < \kappa, X \cap \kappa \in \kappa \text{ and } \mathbf{L} \models \text{“o.t.}(X \cap \vartheta) \text{ is a cardinal”}\}$  is stationary. The notion of the strong reflecting property for  $\mathbf{L}$ -cardinals was introduced before the author knew about the work on  $\nabla_\kappa$  in [14]. The following theorem summarizes the strength of  $\nabla_{\omega_n}$  for  $n \in \omega$ .

**Theorem 2.30** (Räsch & Schindler, [14, Theorems 2 & 4 and Corollary 12]) (1) *The following theories are equiconsistent:*

- (a) ZFC +  $\nabla_{\omega_1}$ .
- (b) ZFC +  $\nabla_{\omega_2}$ .
- (c) ZFC + *there exists a remarkable cardinal.*

(2) *For  $n \geq 3$ ,  $\nabla_{\omega_n}$  is equivalent to  $0^\#$  exists.*

Now we discuss the relationship between  $\text{SRP}^{\mathbf{L}}(\omega_n)$  and  $\nabla_{\omega_n}$  for  $n \in \omega$ . By Theorems 2.23 & 2.30, for  $n \geq 3$ ,  $\text{SRP}^{\mathbf{L}}(\omega_n)$  is equivalent to  $\nabla_{\omega_n}$ . If  $\kappa$  is regular cardinal and  $\nabla_\kappa$  holds, then  $\kappa$  is remarkable in  $\mathbf{L}$  (cf. [14, Lemma 7]). By Proposition 2.9,  $\nabla_{\omega_1}$  implies  $\text{SRP}^{\mathbf{L}}(\omega_1)$  which is strictly weaker. By Theorem 2.17,  $\text{SRP}^{\mathbf{L}}(\omega_2)$  does not imply  $\nabla_{\omega_2}$  since  $\nabla_{\omega_2}$  implies  $\omega_2$  is remarkable in  $\mathbf{L}$ . By Theorems 2.30 & 2.17, the strength of  $\text{SRP}^{\mathbf{L}}(\omega_2)$  is strictly stronger than  $\nabla_{\omega_2}$ .

In Definition 2.2, we only consider countable elementary submodels of  $\mathbf{H}_\kappa$ . Similarly as  $\nabla_\kappa$  we could also consider uncountable elementary submodels of  $\mathbf{H}_\kappa$ . However this does not change the picture. Obviously,  $\text{SRP}^{\mathbf{L}}_{<\omega_1}(\omega_1)$  iff  $\text{SRP}^{\mathbf{L}}(\omega_1)$ . By Proposition 2.10,  $\text{SRP}^{\mathbf{L}}_{<\omega_2}(\omega_2)$  iff  $\text{SRP}^{\mathbf{L}}(\omega_2)$ . By Theorem 2.20, for  $n \geq 3$ ,  $\text{SRP}^{\mathbf{L}}_{<\omega_n}(\omega_n)$  iff  $0^\#$  exists iff  $\text{SRP}^{\mathbf{L}}(\omega_n)$ .

### 3 Harrington’s Principle $\text{HP}(\mathbf{L})$ and its generalization

In this section, we define the generalized Harrington’s Principle  $\text{HP}(M)$  for any inner model  $M$ . Considering various known examples of inner models we give particular characterizations of  $\text{HP}(M)$ , while we also show that in some cases this generalized principle fails.

Recall that for limit ordinal  $\alpha > \omega$ ,  $\alpha$  is  $x$ -admissible if and only if there is no  $\Sigma_1(\mathbf{L}_\alpha[x])$  mapping from an ordinal  $\delta < \alpha$  cofinally into  $\alpha$  (cf. [4, Lemma 7.2]).

**Definition 3.1** Suppose  $M$  is an inner model. The Generalized Harrington’s Principle  $\text{HP}(M)$  denotes the following statement: there is a real  $x$  such that, for any ordinal  $\alpha$ , if  $\alpha$  is  $x$ -admissible then  $\alpha$  is an  $M$ -cardinal, i.e.,  $M \models \alpha$  is a cardinal. We denote Harrington’s Principle by  $\text{HP}(\mathbf{L})$ .

Harrington’s principle  $\text{HP}(\mathbf{L})$  was isolated by Harrington in the proof of his celebrated theorem “ $\text{Det}(\Sigma_1^1)$  implies  $0^\#$ ” in [7].

**Fact 3.2** (Essentially [4];  $Z_4$ ) The model  $\mathbf{L}_{\omega_2}$  has an uncountable set of indiscernibles if and only if  $0^\#$  exists.

<sup>3</sup> Note that  $\mathcal{M}(0^\dagger, \omega, \alpha)$  is the unique transitive  $(0^\dagger, \omega, \alpha)$ -model. For the notation of  $\mathcal{M}(0^\dagger, \omega, \alpha)$ , cf. [10].

**Theorem 3.3** ( $Z_4$ ) *The following are equivalent:*<sup>4</sup>

- (1)  $HP(\mathbf{L})$ .
- (2) *The model  $\mathbf{L}_{\omega_2}$  has an uncountable set of indiscernibles.*
- (3)  $0^\#$  exists.

**Proof.** Note that in  $Z_2$ ,  $0^\#$  implies  $HP(\mathbf{L})$  since any  $0^\#$ -admissible ordinal is an  $\mathbf{L}$ -cardinal. It suffices to show that (1)  $\Rightarrow$  (2). Let  $a$  be the witness real for  $HP(\mathbf{L})$ . We work in  $\mathbf{L}[a]$ . Pick  $\eta > \omega_2$  and  $N$  such that  $\eta$  is  $a$ -admissible,  $N \prec \mathbf{L}_\eta[a]$ ,  $\omega_2 \in N$ ,  $|N| = \omega_1$  and  $N$  is closed under  $\omega$ -sequences. Let  $j : \mathbf{L}_\vartheta[a] \cong N \prec \mathbf{L}_\eta[a]$  be the inverse of the collapsing map and  $\kappa = \text{crit}(j)$ . By  $HP(\mathbf{L})$ ,  $\vartheta$  is an  $\mathbf{L}$ -cardinal. Define  $U = \{X \subseteq \kappa \mid X \in \mathbf{L} \wedge \kappa \in j(X)\}$ . Note that  $(\kappa^+)^{\mathbf{L}} \leq \vartheta < \omega_2$  and  $U \subseteq \mathbf{L}_\vartheta$  is an  $\mathbf{L}$ -ultrafilter on  $\kappa$ . Do the ultrapower construction for  $\langle \mathbf{L}_{\omega_2}, \in, U \rangle$ . Since  $\mathbf{L}_\vartheta[a]$  is closed under  $\omega$ -sequences,  $\mathbf{L}_{\omega_2}/U$  is well founded and hence we get a nontrivial elementary embedding  $e : \mathbf{L}_{\omega_2} \prec \mathbf{L}_{\omega_2}$  with  $\text{crit}(e) = \kappa$ .

Now we show that there exists a club on  $\omega_2$  of regular  $\mathbf{L}$ -cardinals. Suppose  $X \prec \mathbf{L}_\eta[a]$ ,  $\omega_1 \subseteq X$  and  $\omega_2 \in X$ . The transitive collapse of  $X$  is  $\mathbf{L}_{\bar{\eta}}[a]$  for some  $\bar{\eta}$ . Since  $\mathbf{L}_\eta \models \omega_2$  is a regular cardinal,  $\mathbf{L}_{\bar{\eta}} \models \bar{\omega}_2$  is a regular cardinal. By  $HP(\mathbf{L})$ ,  $\bar{\eta}$  is an  $\mathbf{L}$ -cardinal and hence  $\bar{\omega}_2$  is a regular  $\mathbf{L}$ -cardinal. Since  $\omega_1 \subseteq X$ , we have that  $\bar{\omega}_2 = X \cap \omega_2$ . We have shown that if  $X \prec \mathbf{L}_\eta[a]$ ,  $\omega_1 \subseteq X$  and  $\omega_2 \in X$ , then  $X \cap \omega_2 = \bar{\omega}_2$  is a regular  $\mathbf{L}$ -cardinal. So there exists a club on  $\omega_2$  of regular  $\mathbf{L}$ -cardinals. Let  $D$  be such a club such that  $D \cap (\kappa + 1) = \emptyset$ .

**Claim 3.4** *For any  $\alpha \in D$ ,  $e(\alpha) = \alpha$ .*

**Proof.** Suppose  $\alpha \in D$  and  $f \in \mathbf{L}_{\omega_2}$  where  $f : \kappa \rightarrow \alpha$ . Since  $\alpha > \kappa$  is a regular  $\mathbf{L}$ -cardinal,  $f$  is bounded by some  $\eta < \alpha$ . So  $[f] < [c_\eta]$ . Hence  $e(\alpha) = \lim_{\beta \rightarrow \alpha} e(\beta)$ . If  $\beta < \alpha$ , then  $|e(\beta)| \leq (|\beta^\kappa|)^{\mathbf{L}} \leq \alpha$ . So  $e(\alpha) = \alpha$ .  $\square$

We define a sequence  $\langle C_\alpha : \alpha < \omega_1 \rangle$  as follows. Let  $C_0 = D$ . For any  $\nu < \omega_1$ ,  $C_{\nu+1} = \{\mu \in C_\nu \mid \mu \text{ is the } \mu\text{-th element of } C_\nu \text{ in the increasing enumeration of } C_\nu\}$ . If  $\nu \leq \omega_1$  is a limit ordinal,  $C_\nu = \bigcap_{\beta < \nu} C_\beta$ . Note that  $C_\nu$  is a club on  $\omega_2$  for all  $\nu \leq \omega_1$ . By Claim 3.4, for  $\nu \leq \omega_1$ ,  $e \upharpoonright C_\nu = \text{id}$ . Now we will find  $\omega_1$ -many indiscernibles for  $(\mathbf{L}_{\omega_2}, \in)$ . The rest of the argument essentially follows from [8, Theorem 18.20].

For each  $\nu < \omega_1$ , let  $M_\nu$  be the Skolem hull of  $\kappa \cup C_\nu$  in  $\mathbf{L}_{\omega_2}$ . The transitive collapse of  $M_\nu$  is  $\mathbf{L}_{\omega_2}$ . Let  $i_\nu : \mathbf{L}_{\omega_2} \cong M_\nu \prec \mathbf{L}_{\omega_2}$  be the inverse of the collapsing map and  $\kappa_\nu = i_\nu(\kappa)$ . By [8, Lemmas 18.24, 18.25, & 18.26],  $\{\kappa_\nu \mid \nu < \omega_1\}$  is a set of indiscernibles for  $\mathbf{L}_{\omega_2}$ .<sup>5</sup>  $\square$

**Theorem 3.5** (Cheng, [2])  $Z_3 + HP(\mathbf{L})$  *does not imply*  $0^\#$  *exists.*

By a similar argument as in Theorem 3.3 we can show from  $Z_3 + HP(\mathbf{L})$  that there exists a nontrivial elementary embedding  $j : \mathbf{L}_{\omega_1} \prec \mathbf{L}_{\omega_1}$  and there is a club  $C \subseteq \omega_1$  of regular  $\mathbf{L}$ -cardinals. However, by Theorem 3.5, from these we can not prove in  $Z_3$  that  $0^\#$  exists.

Note that Theorem 3.3 still holds if we replace the term “ $\mathbf{L}$ -cardinal” with any large cardinal notion compatible with  $\mathbf{L}$  in the definition of  $HP(\mathbf{L})$ . This is because the Silver indiscernibles can have any large cardinal property compatible with  $\mathbf{L}$ .<sup>6</sup>

A proof of the following statement can be found in [10, Theorem 21.15]:

**Fact 3.6** *The following are equivalent:*

- (1)  $0^\dagger$  exists.
- (2) *For every uncountable cardinal  $\kappa$  there is a  $\kappa$ -model and a double class  $\langle X, Y \rangle$  of indiscernibles for it such that:  $X \subseteq \kappa$  is closed unbounded,  $Y \subseteq \text{Ord} \setminus (\kappa + 1)$  is a closed unbounded class,  $X \cup \{\kappa\} \cup Y$  contains every uncountable cardinal and the Skolem hull of  $X \cup Y$  in the  $\kappa$ -model is again the model.*

<sup>4</sup> In [2], we define  $0^\#$  as the minimal iterable mouse and prove in  $Z_4$  that  $HP(\mathbf{L})$  is equivalent to “ $0^\#$  exists”. Theorem 3.3 proves that these two definitions of  $0^\#$  are equivalent in  $Z_4$ .

<sup>5</sup> Note that the proof of [8, Theorem 18.20], as opposed to the proof of Theorem 3.3 above, is not done in  $Z_4$ .

<sup>6</sup> Examples of large cardinal notions compatible with  $\mathbf{L}$ : inaccessible cardinals, reflecting cardinals, Mahlo cardinals, weakly compact cardinals, indescribable cardinals, unfoldable cardinals, subtle cardinals, ineffable cardinals, 1-iterable cardinals, remarkable cardinals, 2-iterable cardinals, and  $\omega$ -Erdős cardinals.

**Fact 3.7** Suppose that  $A$  is a set,  $X \prec \mathbf{L}_\alpha[A]$  where  $\alpha \in \text{Ord} \cup \{\text{Ord}\}$  and the transitive closure of  $A \cap \mathbf{L}_\alpha[A]$  is contained in  $X$ . Then  $X \cong \mathbf{L}_{\alpha'}[A]$  for some  $\alpha' \leq \alpha$ . [12, Lemma 1.7]

**Fact 3.8** (Folklore) Suppose  $0^\dagger$  exists,  $\mathbf{L}[U]$  is the unique  $\kappa$ -model and  $\langle X, Y \rangle$  is the double class of indiscernibles for  $\mathbf{L}[U]$  as in Fact 3.6. If  $\alpha \leq \kappa$  is  $0^\dagger$ -admissible, then  $X$  is unbounded in  $\alpha$ , and if  $\alpha > \kappa$  is  $0^\dagger$ -admissible, then  $Y$  is unbounded in  $\alpha$ .<sup>7</sup>

**Theorem 3.9** Suppose  $\kappa$  is a measurable cardinal and  $\mathbf{L}[U]$  is the unique  $\kappa$ -model. Then  $\text{HP}(\mathbf{L}[U])$  if and only if  $0^\dagger$  exists.

*Proof.* “ $\Rightarrow$ ”. Let  $x$  be the witness real for  $\text{HP}(\mathbf{L}[U])$ . Pick  $\lambda > 2^\kappa$  and  $X$  such that  $\lambda$  is  $(x, U)$ -admissible,  $X \prec \mathbf{L}_\lambda[U][x]$ ,  $|X| = 2^\kappa$ ,  $X$  is closed under  $\omega$ -sequences and the transitive closure of  $U \cap \mathbf{L}_\lambda[U]$  is contained in  $X$ . By Fact 3.7, the transitive collapse of  $X$  is of the form  $\mathbf{L}_\vartheta[U][x]$ . Let  $j : \mathbf{L}_\vartheta[U][x] \cong X \prec \mathbf{L}_\lambda[U][x]$  be the inverse of the collapsing map and  $\eta = \text{crit}(j)$ . Note that  $\eta > \kappa$ . Since  $\vartheta$  is  $(x, U)$ -admissible, by  $\text{HP}(\mathbf{L}[U])$ ,  $\vartheta$  is an  $\mathbf{L}[U]$ -cardinal. Define  $\bar{U} = \{X \subseteq \eta \mid X \in \mathbf{L}[U] \text{ and } \eta \in j(X)\}$ . Since  $(\eta^+)^{\mathbf{L}[U]} \leq \vartheta$ , we have that  $\bar{U} \subseteq \mathbf{L}_\vartheta[U]$ . Then  $\bar{U}$  is an  $\mathbf{L}[U]$ -ultrafilter on  $\eta$ . Since  $\mathbf{L}_\vartheta[U][x]$  is closed under  $\omega$ -sequences,  $\bar{U}$  is countably complete. So we can build a nontrivial embedding from  $\mathbf{L}[U]$  to  $\mathbf{L}[U]$  with critical point greater than  $\kappa$ . By Fact 2.26,  $0^\dagger$  exists.

“ $\Leftarrow$ ”. Suppose  $0^\dagger$  exists and  $\alpha$  is  $0^\dagger$ -admissible. We show that  $\alpha$  is an  $\mathbf{L}[U]$ -cardinal. By Fact 3.6, let  $\langle X, Y \rangle$  be the double class of indiscernibles for  $\mathbf{L}[U]$ . If  $\alpha \leq \kappa$ , then by Fact 3.8,  $\alpha \in X$ . If  $\alpha > \kappa$ , then by Fact 3.8,  $\alpha \in Y$ . Trivially, elements of  $X$  and  $Y$  are  $\mathbf{L}[U]$ -cardinals.

**Fact 3.10** Suppose there is no inner model with one measurable cardinal and let  $K$  be the corresponding core model. Then,  $K$  has the rigidity property. [13, 16]

**Corollary 3.11** (1) Suppose  $0^\#$  exists. Then  $\text{HP}(\mathbf{L}[0^\#])$  if and only if  $(0^\#)^\#$  exists.

(2) Suppose there is no inner model with one measurable cardinal and that  $K$  is the corresponding core model. Then  $\text{HP}(K)$  does not hold.

*Proof.* (1) follows from the proof of “ $\text{HP}(\mathbf{L}) \Leftrightarrow 0^\#$  exists”. Note that if  $\alpha$  is  $(0^\#)^\#$ -admissible and  $I$  is the class of Silver indiscernibles for  $\mathbf{L}[0^\#]$ , then  $I$  is unbounded in  $\alpha$  and hence  $\alpha \in I$ .

For (2), note that  $K = \mathbf{L}[\mathcal{M}]$  where  $\mathcal{M}$  is a class of mice. Suppose  $\text{HP}(K)$  holds and  $x$  is the witness real for  $\text{HP}(K)$ . Pick  $\vartheta > \omega_2$  and  $X$  such that  $\vartheta$  is  $(\mathcal{M}, x)$ -admissible,  $X \prec \mathbf{J}_\vartheta[\mathcal{M}, x]$ ,  $\omega_2 \in X$ ,  $|X| = \omega_1$  and  $X$  is closed under  $\omega$ -sequences. Since  $K \models \text{GCH}$ , such an  $X$  exists. By the condensation theorem for  $K$ , let  $j : \mathbf{J}_{\vartheta'}[\mathcal{M} \upharpoonright \vartheta', x] \cong X \prec \mathbf{J}_\vartheta[\mathcal{M}, x]$  be the inverse of the collapsing map. Let  $\lambda = \text{crit}(j)$  and  $U = \{X \subseteq \lambda \mid X \in K \text{ and } \lambda \in j(X)\}$ . Note that  $\vartheta'$  is a  $K$ -cardinal and  $U$  is a countably complete  $K$ -ultrafilter on  $\lambda$ . So there is a nontrivial elementary embedding from  $K$  to  $K$  which contradicts Fact 3.10.  $\square$

From proof of Corollary 3.11(2), if  $M$  is an  $\mathbf{L}$ -like inner model,  $M$  has the rigidity property and some proper form of condensation, and  $M \models \text{CH}$ , then  $\text{HP}(M)$  does not hold.

**Fact 3.12** ( $\text{AD}^{\mathbf{L}(\mathbb{R})}$ , [16])  $\text{HOD}^{\mathbf{L}(\mathbb{R})} = \mathbf{L}(P)$  for some  $P \subseteq \Theta$  where  $\Theta = \sup\{\alpha \mid \exists f \in L(R)(f : R \rightarrow \alpha \text{ is surjective})\}$ .

It is an open question whether there exists a nontrivial elementary embedding from  $\text{HOD}$  to  $\text{HOD}$ .<sup>8</sup> However, the following fact shows that the answer to this question is negative for embeddings which are definable in  $V$  from parameters.

**Fact 3.13** (Hamkins, Kirmayer, & Perlmutter, [9, Theorem 35]) Do not assume AC. There is no nontrivial elementary embedding from  $\text{HOD}$  to  $\text{HOD}$  that is definable in  $V$  from parameters.

**Theorem 3.14** ( $\text{ZF} + \text{AD}^{\mathbf{L}(\mathbb{R})}$ )  $\text{HP}(\text{HOD})$  does not hold.

*Proof.* By Fact 3.12, under  $\text{ZF} + \text{AD}^{\mathbf{L}(\mathbb{R})}$ ,  $\text{HOD} = \mathbf{L}(P)$  for some  $P \subseteq \Theta$ . Suppose  $\text{HP}(\text{HOD})$  holds. Then, since  $\mathbf{L}(P) \models \text{CH}$ , by a similar proof as in Corollary 3.11(2) we can show that there exists a nontrivial

<sup>7</sup> We should like to thank W. Hugh Woodin and Sy Friedman for pointing out this fact to us. The proof of this fact is essentially similar as the proof of the following standard fact: if  $0^\#$  exists,  $I$  is the class of Silver indiscernibles and  $\alpha$  is  $0^\#$ -admissible, then  $I$  is unbounded in  $\alpha$  (cf. [5, Theorem 4.3]).

<sup>8</sup> The answer to this question is negative if  $V = \text{HOD}$ . [9, Theorem 21] provides a very easy proof of the Kunen inconsistency in the case  $V = \text{HOD}$ .

elementary embedding  $j : \mathbf{L}(P) \rightarrow \mathbf{L}(P)$ . Note that  $j$  is definable in  $V$  from parameters, i.e., there is a formula  $\varphi$  and parameter  $\vec{a}$  such that  $j(x) = y$  if and only if  $\varphi(x, y, \vec{a})$ . This contradicts Fact 3.13.  $\square$

### 4 Relationship between HP(L) and the strong reflecting property for L-cardinals

In this section, we discuss the relationship between the strong reflecting property for L-cardinals and Harrington's Principle HP(L).

**Theorem 4.1** (Set forcing)  $\text{SRP}^{\mathbf{L}}(\omega_1)$  implies  $\text{Con}(Z_2 + \text{HP}(\mathbf{L}))$ .

**Proof.** Suppose  $\text{SRP}^{\mathbf{L}}(\omega_1)$  holds and we want to build a model of  $Z_2 + \text{HP}(\mathbf{L})$ . By Proposition 2.9,  $\omega_1$  is limit cardinal in  $\mathbf{L}$ . i.e.,  $\{\alpha < \omega_1 \mid \alpha \text{ is an L-cardinal}\}$  is a club. Let  $C = \{\omega \leq \alpha < \omega_1 \mid \alpha \text{ is an L-cardinal and } \mathbf{L}_\alpha \prec \mathbf{L}_{\omega_1}\}$ . Note that  $C$  is a club. Let

$$D = \{\gamma < \omega_1 \mid (\mathbf{L}_\gamma[C], C \cap \gamma) \prec (\mathbf{L}_{\omega_1}[C], C)\}.$$

Note that  $D \subseteq C$ . Define  $F : \omega^\omega \rightarrow \omega^\omega$  as follows: if  $y \subseteq \omega$  codes  $\gamma$ , then  $F(y)$  is a real which codes  $(\beta, C \cap \beta)$  where  $\beta$  is the least element of  $D$  such that  $\beta > \gamma$  (since  $D$  is a club in  $\omega_1$ , such a  $\beta$  exists); if  $y$  does not code an ordinal, let  $F(y) = \emptyset$ .

Let  $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$  be a pairwise almost disjoint set of reals such that  $\delta_\alpha$  is the  $<_{\mathbf{L}[C]}$ -least real which is almost disjoint from any member of  $\{\delta_\beta \mid \beta < \alpha\}$  and  $\langle \delta_\nu \mid \nu < \omega \rangle \in \mathbf{L}_\alpha$  for every admissible ordinal  $\alpha < \omega_1$ .

Let  $\langle x_\alpha \mid \alpha < \omega_1 \rangle$  be the enumeration of  $\wp(\omega)$  in  $\mathbf{L}[C]$  in the order of construction. Let  $Z_F \subseteq \omega_1$  be defined as:

$$Z_F = \{\alpha \cdot \omega + i \mid \alpha < \omega_1 \wedge i \in F(x_\alpha)\}.$$

Now we do almost disjoint forcing to code  $Z_F$  via  $\langle \delta_\alpha \mid \alpha < \omega_1 \rangle$ . Then we get a real  $x$  such that  $\alpha \in Z_F \Leftrightarrow |x \cap \delta_\alpha| < \omega$ . The forcing is c.c.c. and hence preserves all cardinals.

Now we work in  $\mathbf{L}[x]$ . Take the least  $\vartheta$  such that  $\mathbf{L}_\vartheta[x] \models Z_2$ . We will show that  $\mathbf{L}_\vartheta[x] \models \text{HP}(\mathbf{L})$ . By absoluteness, it suffices to show that if  $\alpha < \vartheta$  is  $x$ -admissible, then  $\alpha$  is an L-cardinal. Fix some  $x$ -admissible  $\alpha < \vartheta$  and let  $\gamma_0 = \sup(\alpha \cap D)$ .

If  $\alpha \cap D = \emptyset$ , let  $\gamma_0 = 0$ . Note that if  $\gamma_0 > 0$ , then  $\gamma_0 \in D$ . We assume that  $\gamma_0 < \alpha$  and try to get a contradiction. Let  $\alpha_0$  be the least admissible ordinal such that  $\alpha_0 > \gamma_0$ . Since  $\alpha$  is admissible, we have  $\alpha_0 \leq \alpha$ .

**Claim 4.2** We have that  $C \cap \alpha_0 = C \cap (\gamma_0 + 1)$ .

**Proof.** We show that  $C \cap \alpha_0 \subseteq C \cap (\gamma_0 + 1)$ . Suppose  $\gamma \in C \cap \alpha_0$  and  $\gamma > \gamma_0$ . Since  $\gamma \in C$ , we have that  $\mathbf{L}_\gamma \prec \mathbf{L}_{\omega_1}$ . Since  $\alpha_0$  is definable from  $\gamma_0$ , it follows that  $\alpha_0$  is definable in  $\mathbf{L}_\gamma$ . So  $\alpha_0 \leq \gamma$ . Contradiction.  $\square$

By Claim 4.2,  $\mathbf{L}_{\alpha_0}[C] = \mathbf{L}_{\alpha_0}[C \cap \gamma_0]$ . We need the following lemma to get that  $\mathbf{L}_{\gamma_0}[C \cap \gamma_0][x] = \mathbf{L}_{\gamma_0}[x]$  in Claim 4.5.

**Lemma 4.3** We have that  $C \cap \gamma_0 \in \mathbf{L}_{\gamma_0+1}[x]$ .

**Proof.** We prove by induction that for any  $\gamma \in D \cap \vartheta$ ,  $C \cap \gamma \in \mathbf{L}_{\gamma+1}[x]$ . Fix  $\gamma \in D \cap \vartheta$ . Suppose for any  $\gamma' \in D \cap \gamma$ ,  $C \cap \gamma' \in \mathbf{L}_{\gamma'+1}[x]$ . We show that  $C \cap \gamma \in \mathbf{L}_{\gamma+1}[x]$ .

**Case 1.** There is  $\gamma' \in D$  such that  $\gamma$  is the least element of  $D$  such that  $\gamma > \gamma'$ . Let  $\eta$  be the least admissible ordinal such that  $\eta > \gamma'$ . By a similar argument as in Claim 4.2,  $C \cap \eta = C \cap (\gamma' + 1)$ . From our definitions, for any  $\beta < \eta$  we have: (1)  $\langle x_\xi \mid \xi \in \beta \rangle \in \mathbf{L}_\eta[C] = \mathbf{L}_\eta[C \cap \gamma']$ ; (2)  $\langle \delta_\xi \mid \xi \in \beta \rangle \in \mathbf{L}_\eta[C] = \mathbf{L}_\eta[C \cap \gamma']$ ; and (3)  $\langle x_\xi \mid \xi \in \eta \rangle$  enumerates  $\wp(\omega) \cap \mathbf{L}_\eta[C] = \wp(\omega) \cap \mathbf{L}_\eta[C \cap \gamma']$ .

Suppose  $y \subseteq \omega$  and  $y \in \mathbf{L}_\eta[C \cap \gamma']$ . Then  $y = x_\xi$  for some  $\xi < \eta$ . Note that  $\xi \cdot \omega + i < \eta$  for any  $i < \omega$ . Moreover,  $i \in F(y)$  if and only if  $|x \cap \delta_{\xi \cdot \omega + i}| < \omega$ . So  $F(y) \in \mathbf{L}_\eta[C \cap \gamma'] [x]$ . Hence we have shown that if  $y \in \wp(\omega) \cap \mathbf{L}_\eta[C \cap \gamma']$ , then  $F(y) \in \mathbf{L}_\eta[C \cap \gamma', x]$ .

**Claim 4.4** We have that  $\mathbf{L}_\eta[C \cap \gamma'] \models \gamma' < \omega_1$ .

**Proof.** Suppose, towards a contradiction, that

$$\gamma' = \omega_1^{\mathbf{L}_\eta[C \cap \gamma']}. \tag{1}$$

Let  $\mathbb{P}$  be the almost disjoint forcing that codes  $Z_F$  via the almost disjoint system  $\langle \delta_\beta \mid \beta < \omega_1 \rangle$ , i.e.,  $\mathbb{P} = [\omega]^{<\omega} \times [Z_F]^{<\omega}$  with the order defined by  $(p, q) \leq (p', q')$  iff  $p \supseteq p', q \supseteq q'$  and  $\forall \alpha \in q' (p \cap \delta_\alpha \subseteq p')$ . From our definitions of  $C, F$  and  $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ ,  $\mathbb{P}$  is a definable subset of  $\mathbf{L}_{\omega_1}[C]$ . A standard argument gives that  $\mathbb{P}$  has the  $\omega_1$ -c.c. in  $\mathbf{L}_{\omega_1}[C]$ , i.e., if  $D \subseteq \mathbb{P}$  is a maximal antichain with  $D \in \mathbf{L}_{\omega_1}[C]$ , then  $\mathbf{L}_{\omega_1}[C] \models D$  is at most countable. Let  $\mathbb{P}^* = \mathbb{P} \cap \mathbf{L}_{\gamma'}[C]$ . Since  $\gamma' \in D$ , we have that

$$(\mathbf{L}_{\gamma'}[C], C \cap \gamma') \prec (\mathbf{L}_{\omega_1}[C], C). \tag{2}$$

Suppose  $D^* \subseteq \mathbb{P}^*$  is a maximal antichain with  $D^* \in \mathbf{L}_{\gamma'}[C]$ . Then by (2),  $D^*$  is a maximal antichain in  $\mathbb{P}$ . Since  $\mathbf{L}_{\omega_1}[C] \models D^*$  is at most countable, by (2),  $\mathbf{L}_{\gamma'}[C] \models D^*$  is at most countable. So  $\mathbb{P}^*$  has the  $\omega_1$ -c.c. in  $\mathbf{L}_{\gamma'}[C]$ . By (1),

$$\mathbf{L}_\eta[C \cap \gamma'] \cap 2^\omega = \mathbf{L}_{\gamma'}[C \cap \gamma'] \cap 2^\omega. \tag{3}$$

Since  $\mathbb{P}^*$  has the  $\omega_1$ -c.c. in  $\mathbf{L}_{\gamma'}[C]$ , by (3),  $\mathbb{P}^*$  has the  $\omega_1$ -c.c. in  $\mathbf{L}_\eta[C \cap \gamma']$ .

We show that  $x$  is generic over  $\mathbf{L}_\eta[C \cap \gamma']$  for  $\mathbb{P}^*$ . Let  $Y \subseteq \mathbb{P}^*$  be a maximal antichain with  $Y \in \mathbf{L}_\eta[C \cap \gamma']$ . Since  $\mathbb{P}^*$  has the  $\omega_1$ -c.c. in  $\mathbf{L}_\eta[C \cap \gamma']$ , by (1),  $Y \in \mathbf{L}_{\gamma'}[C \cap \gamma']$ . By (2),  $Y$  is a maximal antichain in  $\mathbb{P}$ . So the filter given by  $x$  meets  $Y$ .

Note that  $\gamma' = \omega_1^{\mathbf{L}_\eta[C \cap \gamma']} = \omega_1^{\mathbf{L}_\eta[C \cap \gamma'][x]}$ . Since  $\gamma' \in D$ , by induction hypothesis  $\mathbf{L}_{\gamma'}[C \cap \gamma', x] = \mathbf{L}_{\gamma'}[x]$ . So  $\mathbf{L}_{\gamma'}[x] \models Z_2$  which contradicts the minimality of  $\vartheta$ . This finishes the proof of Claim 4.4.  $\square$

Take  $y \in \mathbf{L}_\eta[C \cap \gamma'] \cap \wp(\omega)$  such that  $y$  codes  $\gamma'$ . So  $F(y)$  codes  $(\gamma, C \cap \gamma)$  and  $F(y) \in \mathbf{L}_\eta[C \cap \gamma', x]$ . Then  $F(y)$  is definable in  $\mathbf{L}_{\gamma'}[C \cap \gamma', x]$ . By induction hypothesis,  $F(y) \in \mathbf{L}_{\gamma+1}[x]$ . Since  $F(y)$  codes  $C \cap \gamma$ , we have that  $C \cap \gamma \in \mathbf{L}_{\gamma+1}[x]$ .

**Case 2.** The ordinal  $\gamma$  is the least element of  $D$ . Take  $y \in \mathbf{L}_\omega[C] \cap \wp(\omega)$  such that  $y$  codes 0. Then  $y = x_0$ . Since  $\gamma$  is the least element of  $D$  such that  $\gamma > 0$ ,  $F(y)$  codes  $C \cap \gamma$ . Note that for any  $\beta < \omega$ ,  $\langle \delta_\xi \mid \xi \in \beta \rangle \in \mathbf{L}_\omega[C]$  and  $i \in F(y)$  if and only if  $|x \cap \delta_i|$  is finite. So  $F(y)$  is definable in  $\mathbf{L}_\omega[x, C]$ . Since  $C \cap \omega = \emptyset$ , we have that  $F(y) \in \mathbf{L}_{\gamma+1}[x]$ . Since  $F(y)$  codes  $C \cap \gamma$ , we have that  $C \cap \gamma \in \mathbf{L}_{\gamma+1}[x]$ .

**Case 3.** The ordinal  $\gamma$  is a limit point of  $D$ . Then a standard argument gives that  $C \cap \gamma \in \mathbf{L}_{\gamma+1}[x]$  by induction hypothesis.

Since  $\gamma_0 \in D \cap \vartheta$ , we have  $C \cap \gamma_0 \in \mathbf{L}_{\gamma_0+1}[x]$ . This finishes the proof of Lemma 4.3.  $\square$

**Claim 4.5** *The ordinal  $\gamma_0$  is countable in  $\mathbf{L}_{\alpha_0}[C \cap \gamma_0]$ .*

**Proof.** The proof is essentially the same as Claim 4.4 (replace  $\eta$  by  $\alpha_0$  and  $\gamma'$  by  $\gamma_0$ ). Suppose, towards a contradiction, that  $\gamma_0 = \omega_1^{\mathbf{L}_{\alpha_0}[C \cap \gamma_0]}$ . We let  $\mathbb{P}$  be the almost disjoint forcing that codes  $Z_F$  via  $\langle \delta_\beta \mid \beta < \omega_1 \rangle$  as in the proof of Claim 4.4. By the similar argument, we can show that  $x$  is generic over  $\mathbf{L}_{\alpha_0}[C \cap \gamma_0]$  for  $\mathbb{P}^* = \mathbb{P} \cap \mathbf{L}_{\gamma_0}[C]$ . Since  $\gamma_0 = \omega_1^{\mathbf{L}_{\alpha_0}[C \cap \gamma_0]} = \omega_1^{\mathbf{L}_{\alpha_0}[C \cap \gamma_0][x]}$  and by Lemma 4.3,  $\mathbf{L}_{\gamma_0}[C \cap \gamma_0][x] = \mathbf{L}_{\gamma_0}[x]$ , we have  $\mathbf{L}_{\gamma_0}[x] \models Z_2$  which contradicts the minimality of  $\vartheta$ .  $\square$

From our definitions, we have:

$$\text{For } \eta < \alpha_0, \langle \delta_\beta : \beta < \eta \rangle \in \mathbf{L}_{\alpha_0}[C] = \mathbf{L}_{\alpha_0}[C \cap \gamma_0]; \tag{4}$$

$$\langle x_\alpha \mid \alpha < \alpha_0 \rangle \text{ enumerates } \wp(\omega) \cap \mathbf{L}_{\alpha_0}[C] = \wp(\omega) \cap \mathbf{L}_{\alpha_0}[C \cap \gamma_0]. \tag{5}$$

**Claim 4.6** *If  $y \in \wp(\omega) \cap \mathbf{L}_{\alpha_0}[C \cap \gamma_0]$ , then  $F(y) \in \mathbf{L}_{\alpha_0}[x]$ .*

**Proof.** Suppose  $y \in \wp(\omega) \cap \mathbf{L}_{\alpha_0}[C \cap \gamma_0]$ . By (5),  $y = x_\xi$  for some  $\xi < \alpha_0$ . Note that for  $\xi < \alpha_0$ ,  $\xi \cdot \omega + i < \alpha_0$  for any  $i \in \omega$ . By the definition of  $Z_F$ ,  $i \in F(y) \Leftrightarrow \xi \cdot \omega + i \in Z_F \Leftrightarrow |x \cap \delta_{\xi \cdot \omega + i}| < \omega$ . By (4),  $F(y) \in \mathbf{L}_{\alpha_0}[C \cap \gamma_0][x]$ . Since  $C \cap \gamma_0 \in \mathbf{L}_{\gamma_0+1}[x]$  by Lemma 4.3, we have  $\mathbf{L}_{\alpha_0}[C \cap \gamma_0][x] = \mathbf{L}_{\alpha_0}[x]$ . So  $F(y) \in \mathbf{L}_{\alpha_0}[x]$ .  $\square$

By Claim 4.5, there exists a real  $y \in \mathbf{L}_{\alpha_0}[C \cap \gamma_0]$  such that  $y$  codes  $\gamma_0$ . Note that  $F(y)$  codes  $\gamma_1$  where  $\gamma_1$  is the least element of  $C$  such that  $\gamma_1 > \gamma_0$  and  $(\mathbf{L}_{\gamma_1}[C], C \cap \gamma_1) \prec (\mathbf{L}_{\omega_1}[C], C)$ . Since  $F(y)$  codes  $\gamma_1$  and  $F(y) \in \mathbf{L}_{\alpha_0}[x]$ ,

we have  $\gamma_1 < \alpha_0$ . Since  $\gamma_1 < \alpha$  and  $(\mathbf{L}_{\gamma_1}[C], C \cap \gamma_1) \prec (\mathbf{L}_{\omega_1}[C], C)$ , by the definition of  $\gamma_0$ , we have that  $\gamma_1 \leq \gamma_0$ . Contradiction.

So the assumption  $\gamma_0 < \alpha$  is false. Then  $\gamma_0 = \alpha$ . So  $\alpha \in C$  and hence  $\alpha$  is an  $\mathbf{L}$ -cardinal. We have shown that  $\mathbf{L}_{\vartheta}[x] \models Z_2 + \text{HP}(\mathbf{L})$ .  $\square$

**Theorem 4.7** (Cheng, [2, Theorems 3.1 & 3.2]; class forcing)  $Z_2 + \text{HP}(\mathbf{L})$  is equiconsistent with ZFC and  $Z_3 + \text{HP}(\mathbf{L})$  is equiconsistent with ZFC + there exists a remarkable cardinal.

**Corollary 4.8** (a) For  $n \geq 3$ ,  $\text{SRP}^{\mathbf{L}}(\omega_n)$  is equivalent to  $\text{HP}(\mathbf{L})$ .

(b)  $\text{SRP}^{\mathbf{L}}(\omega_2)$  is strictly stronger than  $Z_3 + \text{HP}(\mathbf{L})$ .

(c)  $\text{SRP}^{\mathbf{L}}(\omega_1)$  is strictly stronger than  $Z_2 + \text{HP}(\mathbf{L})$ .

**Proof.** (a) follows from Theorems 2.23 & 3.3. (b) follows from Theorems 2.17 & 4.7. (c) follows from Theorems 4.1 & 4.7 and Proposition 2.9. Items (b) and (c) use set forcing.

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