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The Relative Zero Operator: A Causal Nonlocal Operator with Memory in Functional and Spectral Analysis

Author

Marcelo Barboza Duarte

Independent Researcher Rio de Janeiro, Brazil

ORCID: 0000-0002-5948-5714

Email: duartem1900@gmail.com

Abstract

We present a rigorous functional-analytic and spectral study of the Relative Zero Operator, a causal nonlocal operator defined as the difference between a function and its kernel-based causal average over a finite memory interval. The operator arises naturally in nonlocal effective field theories with memory, where physical or mathematical fields are regulated relative to a dynamically defined reference rather than an absolute baseline.

Working in standard Banach and Hilbert spaces, we establish linearity, boundedness, and continuity under minimal assumptions on the memory kernel. We characterize the null space as the set of functions invariant under causal averaging and analyze the fixed-point structure and stability under perturbations. In translation-invariant settings, we derive the Fourier-domain representation of the operator and show that its spectrum is determined by the kernel transform, leading to a systematic suppression of low-frequency modes and a high-pass filtering behavior.

We discuss conditions for closed range and non-invertibility, emphasizing the role of kernel normalization and memory scale. Explicit illustrative examples are provided for representative kernels. The results establish the Relative Zero Operator as a well-defined nonlocal operator with causal memory, suitable for applications in effective field theories, nonlocal dynamics, and related areas of mathematical physics.

Keywords

Nonlocal Operators; Causal Memory; Convolution Operators; Functional Analysis; Operator Spectrum; Fixed Points; Banach Spaces; Hilbert Spaces; Kernel Methods; Effective Field Theory

1. Introduction

Nonlocal operators defined through convolution-type averages arise naturally in a wide class of problems in mathematical physics, effective field theory, and systems with memory. In such contexts, the relevant dynamical quantities are often defined relative to a reference determined by past values of the field, rather than with respect to an absolute baseline. This motivates the study of operators that subtract a causal average from a given function, thereby isolating deviations from a dynamically defined reference state.

The Relative Zero Operator is defined precisely in this manner. Given a causal averaging operator constructed from a normalized memory kernel over a finite interval, the Relative Zero Operator acts as the complement of this averaging map. From a mathematical standpoint, it belongs to the class of bounded linear operators generated by convolution with integrable kernels and their differences from the identity.

The aim of the present work is to analyze this operator as an object of functional analysis and operator theory, independently of any specific physical realization. We establish its basic properties in standard Banach and Hilbert spaces, characterize its kernel and fixed-point structure, and study its spectral behavior in translation-invariant settings. Particular attention is given to the role of kernel normalization and memory scale in determining stability, non-invertibility, and frequency-selective behavior.

The paper is organized as follows. In Section 2 we introduce notation and define the operator in general functional spaces. Section 3 establishes linearity, boundedness, and continuity. Section 4 analyzes the kernel and range structure. Fixed points and stability under perturbations are discussed in Section 5. Section 6 is devoted to the Fourier-domain representation and spectral properties. Asymptotic behavior and illustrative examples are presented in Sections 7 and 8, followed by a discussion and concluding remarks.

2. Preliminaries and Notation

Let Xbe a Banach space of real- or complex-valued functions defined on $\mathbb{R}$, such as

$$L^p(\mathbb{R}), 1 \leq p \leq \infty,$$

spaces of continuous functions $C([a,b])$, or Sobolev spaces $W^{k,p}(\mathbb{R})$, whenever convolution with integrable kernels is well defined and continuous.

Let $\tau > 0$denote a fixed memory scale. We consider a memory kernel

$$w: [0,\tau] \to \mathbb{R}$$

satisfying the following assumptions:

1. $w(u) \geq 0$for almost every $u \in [0,\tau]$,
2. $\int_0^\tau \quad w(u)\, du = 1$,
3. $w \in L^1([0,\tau])$.

These conditions ensure that $w$defines a normalized causal averaging kernel.

2.1 Causal Averaging Operator

We define the causal averaging operator $A: X \to X$ by

$$(Af)(t) = \int_0^\tau \quad f(t-u)\, w(u)\, du,$$

whenever the integral is well defined. In translation-invariant spaces such as $L^p(\mathbb{R})$, the operator A is a convolution with an L^1 kernel and is therefore bounded.

2.2 Definition of the Relative Zero Operator

The Relative Zero Operator $Z: X \to X$ is defined by

$$(Zf)(t) = f(t) - (Af)(t).$$

By construction, Z measures the deviation of a function from its causal kernel average. The operator depends explicitly on the choice of kernel w and the memory scale τ, and it is nonlocal in time due to the finite integration interval.

Throughout the paper, we denote by $\hat{f}(\omega)$ the Fourier transform of f whenever $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and by $\widehat{w}(\omega)$ the Fourier transform of the kernel extended by zero outside $[0, \tau]$.

3. Basic Properties of the Relative Zero Operator

3.1 Linearity

Proposition 3.1. The Relative Zero Operator Z is linear on any vector space X where the causal averaging operator A is well defined.

Proof. Linearity follows immediately from the linearity of the integral defining A. For any $f, g \in X$ and scalars α, β,

$$Z(\alpha f + \beta g) = \alpha Z f + \beta Z g.$$

□

3.2 Boundedness

Theorem 3.2. Let $X = L^p(\mathbb{R})$ with $1 \le p \le \infty$. Then the operator $Z: X \to X$ is bounded.

Proof. Since A is convolution with an L^1 kernel,

$$\| Af \|_p \le \| w \|_1 \| f \|_p = \| f \|_p.$$

Therefore,

$$\| Zf \|_p \le \| f \|_p + \| Af \|_p \le 2 \| f \|_p,$$

which proves boundedness. □

3.3 Continuity

Proposition 3.3. If X is a Banach space stable under translation and convolution with L^1 kernels, then Z is continuous.

Proof. Continuity follows from boundedness of both the identity operator and the averaging operator A. □

3.4 Non-Idempotency

The Relative Zero Operator is generally **not idempotent**, i.e.

$$Z^2 \neq Z.$$

This follows from the fact that

$$Z^2 = I - 2A + A^2,$$

and $A^2 \neq A$ for generic kernels. This property distinguishes Z from projection operators and reflects the recursive nature of causal averaging when applied iteratively.

4. Kernel and Range Structure

4.1 Kernel of the Operator

Theorem 4.1. The kernel of Z is given by

$$\ker(Z) = \{f \in X : f = Af\}.$$

Proof. By definition, $Zf = 0$ if and only if $f = Af$. □

Corollary 4.1. If f is continuous and invariant under causal averaging with a normalized kernel, then f is constant.

Remark. This result follows from standard averaging arguments: invariance under convolution with a normalized kernel implies that f cannot exhibit local variation on scales larger than the support of the kernel. In this sense, constant functions define the natural null modes of the operator.

4.2 Range and Closedness

The structure of the range of $Z = I - A$ depends sensitively on the spectral properties of the averaging operator A.

Proposition 4.2. If A is a compact operator on a Banach space X and 1 does not belong to the approximate point spectrum of A, then the range of Z is closed.

Remark. This condition is satisfied, for example, when Aacts on spaces of continuous functions over compact intervals with sufficiently regular kernels. In general, closedness of the range cannot be guaranteed without additional spectral assumptions.

5. Fixed Points and Stability

5.1 Fixed Points

We first characterize the fixed points of the Relative Zero Operator.

Proposition 5.1. A function $f \in X$is a fixed point of Zif and only if

$$Af = 0.$$

Proof. By definition, $Zf = f$if and only if

$$f - Af = f,$$

which is equivalent to $Af = 0$. □

Corollary 5.1. If $f \in X$satisfies $Af = 0$and $\widehat{w}(\omega) \neq 0$almost everywhere, then $f = 0$.

Proof. Taking Fourier transforms in translation-invariant spaces,

$$\widehat{Af}(\omega) = \widehat{w}(\omega)\hat{f}(\omega).$$

If $Af = 0$and $\widehat{w}(\omega) \neq 0$almost everywhere, it follows that $\hat{f}(\omega) = 0$, hence $f = 0$. □

Remark. Without additional assumptions on the kernel or the function space, nontrivial solutions of $Af = 0$may exist. The above corollary shows that triviality of fixed points is guaranteed under mild spectral conditions on the kernel.

5.2 Stability Under Perturbations

Proposition 5.2. Let $f_n \to f$in X. Then

$$Zf_n \to Zf$$

in X.

Proof. This follows directly from the continuity of Zestablished in Section 3. □

Interpretation. The Relative Zero Operator is stable under small perturbations of the input function. This property is essential for applications in effective theories, where fields are typically subject to fluctuations or approximation errors.

6. Spectral Analysis

We now analyze the spectral properties of the Relative Zero Operator in translation-invariant settings.

6.1 Fourier Representation

Let $X = L^2(\mathbb{R})$. Extending the kernel w by zero outside $[0, \tau]$, the averaging operator A acts as convolution, and its Fourier transform is given by

$$\widehat{Af}(\omega) = \widehat{w}(\omega)\hat{f}(\omega).$$

Theorem 6.1 (Fourier Symbol). The Fourier symbol of the Relative Zero Operator is

$$\hat{Z}(\omega) = 1 - \widehat{w}(\omega).$$

Proof. Immediate from linearity of the Fourier transform. □

6.2 Spectrum

Theorem 6.2. The spectrum of Z in $L^2(\mathbb{R})$ satisfies

$$\sigma(Z) \subset \{1 - \widehat{w}(\overline{\omega)} : \omega \in \mathbb{R}\}.$$

Remark. For sufficiently regular kernels, the spectrum coincides with the closure of the image of the symbol. The precise spectral type (point, continuous, or residual) depends on the properties of $\widehat{w}$.

6.3 Non-Invertibility

Theorem 6.3. The Relative Zero Operator Z is not invertible on any space containing the constant functions.

Proof. Since w is normalized,

$$Af = f$$

for any constant function f. Hence constants belong to $\ker(Z)$, and Z is not injective. Therefore, Z is not invertible. □

Remark. This non-invertibility is structural and independent of the detailed form of the kernel, provided normalization holds.

6.4 Frequency-Selective Behavior

For normalized kernels, one has

$$\widehat{w}(0) = 1,$$

implying

$$\hat{Z}(0) = 0.$$

Thus, low-frequency modes are suppressed. For large $|\omega|$, $\hat{w}(\omega) \to 0$ for sufficiently regular kernels, yielding

$$\hat{Z}(\omega) \to 1.$$

Interpretation. The Relative Zero Operator acts as a kernel-dependent high-pass filter, suppressing slowly varying components while preserving rapidly varying modes.

7. Asymptotic Behavior

We analyze the behavior of the Relative Zero Operator in the regime where the input function varies slowly compared to the memory scale τ.

Proposition 7.1. Let $f \in X$ be sufficiently smooth and slowly varying on scales larger than τ. Then

$$(Zf)(t) \to 0$$

uniformly as the characteristic variation scale of f becomes much larger than τ.

Proof. For slowly varying functions, one may expand

$$f(t-u) = f(t) - uf'(t) + \mathcal{O}(u^2).$$

Using normalization of the kernel,

$$(Af)(t) = f(t) - \langle u \rangle f'(t) + \mathcal{O}(\tau^2),$$

where $\langle u \rangle = \int_0^\tau \quad uw(u)\, du$. Hence,

$$(Zf)(t) = f(t) - (Af)(t) = \mathcal{O}(\tau),$$

which vanishes in the slow-variation limit. □

Interpretation. The operator suppresses modes that are approximately constant over the memory window, reinforcing its role as a deviation extractor relative to a causal reference.

8. Illustrative Examples

To illustrate the action of the Relative Zero Operator, we consider a simple uniform kernel

$$w(u) = \frac{1}{\tau}, 0 \leq u \leq \tau.$$

Example 1: Constant Function

Let $f(t) = c$, with $c$constant. Then

$$(Af)(t) = c, (Zf)(t) = 0.$$

This confirms that constant functions lie in the kernel of Z.

Example 2: Step Function

Let

$$f(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

A direct computation yields

$$(Zf)(t) = \begin{cases} 0, & t < 0, \\ 1 - \frac{t}{\tau}, & 0 \leq t \leq \tau, \\ 0, & t > \tau. \end{cases}$$

The operator smooths the discontinuity over the memory interval.

Example 3: Sinusoidal Function

Let $f(t) = \sin(\omega t)$. Then

$$\hat{Z}(\omega) = 1 - \widehat{w}(\omega).$$

Low-frequency modes ($\omega\tau \ll 1$) are suppressed, while high-frequency modes ($\omega\tau \gg 1$) are preserved, illustrating the frequency-selective nature of the operator.

9. Discussion

The Relative Zero Operator defines a well-posed bounded linear operator in standard functional spaces, characterized by causal nonlocality and kernel-dependent memory effects. Its mathematical structure is governed by the properties of the averaging kernel, particularly normalization and regularity, which determine the null space, spectral behavior, and stability.

From an operator-theoretic perspective, $Z$is neither a projection nor an invertible map, but rather a deviation operator that removes kernel-invariant components. Its Fourier representation reveals a systematic suppression of low-frequency modes, placing it within a class of nonlocal operators with intrinsic scale selectivity.

These properties make the operator suitable for applications in nonlocal effective theories, systems with memory, and related areas of mathematical physics where relational or dynamically defined reference states play a central role.

10. Conclusion

We have presented a rigorous functional-analytic and spectral analysis of the Relative Zero Operator, a causal nonlocal operator defined as the complement of a kernel-based averaging map. Under minimal assumptions on the memory kernel, the operator is linear, bounded, continuous, and structurally non-invertible due to the presence of kernel-invariant modes.

The Fourier-domain representation shows that the operator acts as a kernel-dependent high-pass filter, suppressing slowly varying components while preserving rapidly varying ones. These results establish the Relative Zero Operator as a mathematically well-defined tool for the analysis of nonlocal systems with causal memory, providing a solid foundation for further developments in effective field theory and related frameworks.

Declarations

Declaration on the Use of Large Language Models

I, the undersigned author, hereby declare that Large Language Models (LLMs) were used exclusively for non-generative editorial assistance during the preparation of this manuscript. Such assistance was limited to improving clarity, grammar, style, and organization of text originally written by the author. No scientific content, theoretical development, equations, derivations, numerical results, interpretations, or conclusions were generated by LLMs. All conceptual ideas, physical arguments, mathematical formulations, computational implementations, data analyses, and scientific interpretations presented in this work are entirely the responsibility of the author. The final version of the manuscript was fully reviewed, verified, and approved by the author.

Author: Marcelo Barboza Duarte
ORCID: 0000-0002-5948-5714

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References

Bekenstein, J. D. Black Holes and Entropy. Phys. Rev. D 7, 2333–2346 (1973).

Belot, G. Geometric Possibility. Oxford University Press (2011).

Botelho, G., Pellegrino, D. & Teixeira, E. Fundamentos de Análise Funcional. Sociedade Brasileira de Matemática (2015).

Boyd, R. W. Nonlinear Optics. Academic Press (2008).

Conway, J. B. A Course in Operator Theory. American Mathematical Society (2000).

Einstein, A. Die Grundlage der allgemeinen Relativitätstheorie. Ann. Phys. 354, 769–822 (1916).

Goldstein, H., Poole, C. P. & Safko, J. L. Classical Mechanics. Pearson (2001).

Hönig, C. S. Análise Funcional e Aplicações. Edusp (1970).

Iório Jr., R. J. & Iório, V. M. Equações Diferenciais Parciais: Uma Introdução. IMPA (2010).

Itzykson, C. & Zuber, J. B. Quantum Field Theory. McGraw-Hill (1980).

Jackson, J. D. Classical Electrodynamics. Wiley (1998).

Khalil, M. & Weltman, A. Effective field theories with memory effects in gravity. Phys. Rev. D 108, 044021 (2023).

Kreyszig, E. Introductory Functional Analysis with Applications. Wiley (1989).

Landau, L. D. & Lifshitz, E. M. Electrodynamics of Continuous Media. Pergamon Press (1984).

Lopes, A. O. Tópicos em Teoria de Operadores. UFRGS (2003).

Melo, S. T. C*-Álgebras e Geometria Não-Comutativa. IMPA (2012).

Maggiore, M. & Mancarella, M. Nonlocal gravity and dark energy. Phys. Rev. D 90, 023005 (2014).

Murphy, G. J. C*-Algebras and Operator Theory. Academic Press (1990).

Nussenzveig, H. M. Curso de Física Básica 3: Eletromagnetismo. Blucher (2015).

Oliveira, C. R. Introdução à Análise Funcional. IMPA (2015).

Peskin, M. E. & Schroeder, D. V. An Introduction to Quantum Field Theory. Westview Press (1995).

Reed, M. & Simon, B. Methods of Modern Mathematical Physics, Vol. I: Functional Analysis. Academic Press (1980).

Salinas, S. R. A. Introdução à Física Estatística. Edusp (2013).

Weinberg, S. The Quantum Theory of Fields, Vol. I. Cambridge University Press (1995).

Woodard, R. P. Nonlocal Models of Cosmic Acceleration. Found. Phys. 44, 213–233 (2014).

Zangwill, A. Modern Electrodynamics. Cambridge University Press (2013).