GUESS THE NUMBER?

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Following Hamkins's analysis of infinitary versions of familiar guessing games [1], it is natural to ask how the strategic structure of finite information games changes when the space of possibilities becomes infinite. In finite settings, games such as *Guess Who?* are governed by straightforward elimination strategies: each question refines a finite domain of candidates, and termination is guaranteed. By contrast, in infinite settings, refinement need not lead to convergence, and the possibility of indefinite play introduces new strategic considerations.

In this spirit, this paper introduces an infinitary variant of *Guess Who?* which we shall call *Guess the Number?*. Suppose we have a game with three players, each secretly choosing a number from the countably infinite set of natural numbers. In the first phase, each player attempts to guess another player's number: α guesses β, β guesses γ, and γ guesses α. When a player's number is correctly guessed, that player is eliminated, and the remaining players advance to a second phase. The game is then played under the rules of *misère*: the eliminator in the first round moves second, and the first player whose number is correctly guessed wins.

Consider the first phase of the game. As opposed to a question in natural language (e.g., "Is your number interesting?"), each player speaks in the language of Presburger arithmetic $\mathcal{L} = \{0, S, +, <\}$ and offers a predicate $P(x)$ where x is the only free variable in P. Let α, β, and γ denote the first, second, and third players, respectively, and let c be a choice function such that $c : \{\alpha, \beta, \gamma\} \to \mathbb{N}$.[1] As a matter of convention, let $P_{(\alpha,m)}$ denote the query α has for β's number which is associated with index m and so forth; let an overline over a number (e.g., $\bar{n}$) denote the expression $\underbrace{S \ldots S}_{n \text{ times}}(0)$ in $\mathcal{L}$.

Definition 0.1 (Sequences of Predicates)**.** Let s be a function from $\mathbb{N}$ to the set of well-formed formulas in $\mathcal{L}$ such that

$$s(n) = \begin{cases} P_{(\alpha,n)}(x) & \text{if } n \bmod 3 = 0, \\ P_{(\beta,n)}(x) & \text{if } n \bmod 3 = 1, \\ P_{(\gamma,n)}(x) & \text{if } n \bmod 3 = 2. \end{cases}$$

Definition 0.2 (Target Function)**.** Let χ be a target singleton function such that

$$\chi(n) = \begin{cases} \{c(\beta)\} & \text{if } n \bmod 3 = 0, \\ \{c(\gamma)\} & \text{if } n \bmod 3 = 1, \\ \{c(\alpha)\} & \text{if } n \bmod 3 = 2. \end{cases}$$

Definition 0.3 (Response Function)**.** Let r be a response function for each query such that

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[1] To be sure, 0 is a natural number.

$$r(n) = \begin{cases} 1 & \text{if } \chi(n) \subseteq \{k \,|\, \mathfrak{N} \models s(n)[\![k]\!]\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{N}$ is the standard model of arithmetic.

Aside from χ, the outputs of all functions are public knowledge for all players.

Definition 0.4 (Characteristic Function)**.** Let t be a function from $\{\alpha, \beta, \gamma\}$ to $\{0, 1, 2\}$ such that

$$t(\kappa) = \begin{cases} 0 & \text{if } \kappa = \alpha, \\ 1 & \text{if } \kappa = \beta, \\ 2 & \text{if } \kappa = \gamma. \end{cases}$$

Definition 0.5 (Learning Function)**.** Let L denote the learning function such that for all $\kappa \in \{\alpha, \beta, \gamma\}$

$$L(\kappa, 0) = \begin{cases} \mathbb{N} \cap \{k \,|\, \mathfrak{N} \models s(t(\kappa))[\![k]\!]\} & \text{if } r(t(\kappa)) = 1, \\ \mathbb{N} \setminus \{k \,|\, \mathfrak{N} \models s(t(\kappa))[\![k]\!]\} & \text{otherwise.} \end{cases}$$

$$L(\kappa, n+1) = \begin{cases} L(\kappa, n) \cap \{k \,|\, \mathfrak{N} \models s(3(n+1) + t(\kappa))[\![k]\!]\} & \text{if } r(3(n+1) + t(\kappa)) = 1, \\ L(\kappa, n) \setminus \{k \,|\, \mathfrak{N} \models s(3(n+1) + t(\kappa))[\![k]\!]\} & \text{otherwise.} \end{cases}$$

Definition 0.6 (Halting Number)**.** H is defined as a halting number iff

$$\exists \sigma((L(\sigma, H) = \chi(t(\sigma))) \wedge \forall j \forall \sigma'(j < H \to L(\sigma', j) \neq \chi(t(\sigma')))).$$

Axiom 0.7 (Axiom of Learning)**.** For all $\kappa \in \{\alpha, \beta, \gamma\}$,

$$\forall n(|L(\kappa, n+1)| > 1 \to L(\kappa, n+1) \subsetneq L(\kappa, n)).$$

Definition 0.8 (1^{st} Win)**.** We say that κ wins the first phase of the game at halting number H iff

$$L(\kappa, H) = \chi(t(\kappa)) \wedge \forall \sigma(t(\sigma) < t(\kappa) \to L(\sigma, H) \neq \chi(t(\sigma))).$$

On one hand, every agent is aiming to find, as quickly as possible, the hidden number of their successor. But they also want to be in a position where the domain of their predecessor's bid on their number is finite. By Axiom 0.7, every query must result in greater specificity. Hence, finiteness is a necessary condition for the player whose number is being guessed to win in the second phase of the game. Suppose α wins the first phase of the game at some halting number H_α, where γ is the survivor while also being the predecessor of α. We say that α's circumstances are ideal for phase 2 iff the following holds:

$$1 < |L(\gamma, H_\alpha - 1)| < \aleph_0 \text{ and } |L(\gamma, H_\alpha - 1)| \leq |L(\beta, H_\alpha - 1)| \leq \aleph_0$$

and so forth for the other players.[2]

[2]To be sure, each player would like the bid size on their own number to be finite though > 1. They would also like the bid size on their competitor's number to be relatively larger during the second phase of the game.

This appears to be a paradoxical situation; α is almost guaranteed to win in the second phase of the game if she is almost guaranteed to lose in the first. Moreover, the naive strategy of exponentially searching for the upper bound of a hidden number and then proceeding with a binary search does not guarantee a win for the whole game. If a player does not have enough opportunity to reveal sufficient information about their own number to their predecessor to force finiteness, the second phase of the game could result in an infinite stall.

Question 0.9. Is there an optimal move for any given player?

As a general rule, it is obvious that each player should avoid situations in which their predecessor guesses poorly while being *close* to eliminating their successor. If a predecessor makes a bid that leaves an infinite subset of $\mathbb{N}$, the player should construct their predicate so that it does not force the successor's domain to become finite. Suppose the player and the successor advance to the next phase and finiteness were forced on the successor; then the player is guaranteed to lose (since they will eventually be forced to enumerate their successor's number); likewise, if the player succeeds in forcing finiteness on her successor and eliminates her, the predecessor advances to the next stage and can stall the game indefinitely by enumerating an infinite sequence of divisibility predicates (e.g., $\exists m(x = \underbrace{m + \ldots + m}_{p_k \text{ times}})$, where p_k is the k^{th} prime number.) Conversely, if the predecessor is a skilled guesser who can force finiteness, the player should focus on discovering the successor's hidden number aggressively unless their successor can act faster by eliminating their predecessor. If the latter scenario does not hold, then the player and the predecessor advance to the next stage where the latter could be forced to enumerate the player's number. Suppose that the latter case is true; then the player should stall by enumerating an infinite sequence of divisibility predicates, which gives them an advantage in the second stage of the game.

How should we go about formalizing this line of reasoning? Let us first introduce a function for each $\kappa \in \{\alpha, \beta, \gamma\}$.

Definition 0.10 (Partial s functions)**.**

$$s_\kappa(n) = \begin{cases} P_{(\kappa,n)}(x) & \text{if } n \bmod 3 = t(\kappa), \\ \uparrow & \text{otherwise.} \end{cases}$$

We write $s(c) =\uparrow$ to indicate that the value of s is undefined when it assumes the value c; otherwise, we write $s(c) =\downarrow$.

Example 0.11 (Naive Strategy)**.** Given a partial s function for κ, an example of a naive strategy is as follows

$$P^{\mathsf{NAIVE}}_{(\kappa,t(\kappa))}(x) = (x < \overline{2^0}),$$

$$P^{\mathsf{NAIVE}}_{(\kappa,3(n+1)+t(\kappa))}(x) = \begin{cases} (x < \overline{2^{n+1}}) & \text{if } |L(\kappa,n)| = \aleph_0, \\ (x < \overline{\lceil \frac{\sup(L(\kappa,n)) + \min(L(\kappa,n))}{2} \rceil}) & \text{otherwise.} \end{cases}$$

Definition 0.12. Let m be a string, then

$$\mathbf{m}_0 = (m + m),$$

$$\mathbf{m}_{n+1} = (\mathbf{m}_n + m).$$

Example 0.13 (Pure Stalling Strategy).

$$P^{\mathsf{STALL}}_{(\kappa,t(\kappa))}(x) = \exists m(x = \mathbf{m}_0)$$

$$P^{\mathsf{STALL}}_{(\kappa,3(n+1)+t(\kappa))}(x) = \begin{cases} \exists m(x = \mathbf{m}_{n+1}) & \text{if } |L(\kappa, n)| = \aleph_0, \\ (x = \overline{\min(L(\kappa, n))}) & \text{otherwise.} \end{cases}$$

Example 0.14 (Mixed Strategies). We offer three mixed strategies for each player:

α)

$$P^{\mathsf{MIXED}}_{(\alpha,0)}(x) = P^{\mathsf{STALL}}_{(\alpha,0)}(x),$$

$$P^{\mathsf{MIXED}}_{(\alpha,3n+3)}(x) = \begin{cases} P^{\mathsf{STALL}}_{(\alpha,3n+3)}(x) & \text{if } |L(\gamma, n)| = \aleph_0 \\ & \quad \text{or } |L(\beta, n)| < \min\big(\{|L(\gamma, n)|,\ |L(\alpha, n)|\}\big), \\ P^{\mathsf{NAIVE}}_{(\alpha,3n+3)}(x) & \text{otherwise.} \end{cases}$$

β)

$$P^{\mathsf{MIXED}}_{(\beta,1)}(x) = \begin{cases} P^{\mathsf{STALL}}_{(\beta,1)}(x) & \text{if } |L(\alpha, 0)| = \aleph_0, \\ P^{\mathsf{NAIVE}}_{(\beta,1)}(x) & \text{otherwise,} \end{cases}$$

$$P^{\mathsf{MIXED}}_{(\beta,3n+4)}(x) = \begin{cases} P^{\mathsf{STALL}}_{(\beta,3n+4)}(x) & \text{if } |L(\alpha, n+1)| = \aleph_0 \\ & \quad \text{or } |L(\gamma, n)| < \min\big(\{|L(\beta, n)|,\ |L(\alpha, n+1)|\}\big), \\ P^{\mathsf{NAIVE}}_{(\beta,3n+4)}(x) & \text{otherwise.} \end{cases}$$

γ)

$$P^{\mathsf{MIXED}}_{(\gamma,2)}(x) = \begin{cases} P^{\mathsf{STALL}}_{(\gamma,2)}(x) & \text{if } |L(\beta, 0)| = \aleph_0 \\ & \quad \text{or } |L(\alpha, 0)| < |L(\beta, 0)|, \\ P^{\mathsf{NAIVE}}_{(\gamma,2)}(x) & \text{otherwise,} \end{cases}$$

$$P^{\mathsf{MIXED}}_{(\gamma,3n+5)}(x) = \begin{cases} P^{\mathsf{STALL}}_{(\gamma,3n+5)}(x) & \text{if } |L(\beta, n+1)| = \aleph_0 \\ & \quad \text{or } |L(\alpha, n+1)| < \min\big(\{|L(\beta, n+1)|,\ |L(\gamma, n)|\}\big), \\ P^{\mathsf{NAIVE}}_{(\gamma,3n+5)}(x) & \text{otherwise.} \end{cases}$$

Previously, we have used the term "strategy" quite loosely, as a method of recursively defining the $3(n+1) + t(\cdot)^{\text{th}}$ predicate in terms of $L(\cdot, n)$. It is now worth re-examining this term in light of the fact that the usual definition of a "winning strategy" does not seem to apply to our game. In standard guessing games such as *Mastermind* or *Wordle*, the roles of the code-maker and the code-breaker are immutable. As such, a winning strategy for the code-breaker is one that eventually results in the codeword being revealed after finitely many guesses. This is not the case for *Guess the Number?*, where each player assumes both the role of a code-maker *and* a code-breaker, and where the ultimate objective is not necessarily equivalent to (and is indeed at times in conflict with) correctly guessing the code-number of one's successor.

Because of this, we may instead turn to the weaker notion of "optimality". We could say that one strategy is more optimal than another if it allows the player to avoid certain undesirable and preventable dangers. To be sure, this is a crude notion that is worth formalizing and being made precise; reflecting on our prior motivations for constructing $P^{\mathsf{MIXED}}_{(\cdot,\cdot)}$, we wanted to avoid the self-inflicted danger of being able to guess one's successor's number without giving our predecessor enough opportunity to "force finiteness", as it were, which is a necessary condition for winning in the second phase of the game. But it is also worth asking if the cure is at least as bad as the disease; although we have tried to avoid stalling during the second phase of the game, if all players employed mixed strategies, they will stall indefinitely in the first. It is certainly ironic that in trying to evade one self-inflicted danger, we have inadvertently stepped into another.

References

[1] Joel David Hamkins. Infinite wordle and the mastermind numbers. *Mathematical Logic Quarterly*, 2023. doi: 10.1002/malq.202200049. URL https://doi.org/10.1002/malq.202200049.

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