From Particle Horizon to Solution Space Selection of the Wheeler-DeWitt Equation: A Mathematical Physics Proof Based on Quantum Mechanics Postulates and Cosmological Boundary Conditions

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The Wheeler-DeWitt equation, as the central equation of quantum gravity, suffers from an excessively large solution space due to the lack of well-defined boundary conditions, leading to a loss of predictive power. Traditional approaches introduce artificial boundary conditions or topological constraints, but they lack support from first principles. This paper demonstrates that, by combining the fundamental postulates of quantum mechanics with the cosmological particle horizon, the physically relevant solutions of the Wheeler-DeWitt equation must belong to the space of bandlimited functions. This selection is entirely based on the principle of observability and requires no presupposition of a transcendent external observer.

INTRODUCTION

The Wheeler-DeWitt equation (DeWitt 1967), as the core equation of quantum gravity, suffers from an excessively large solution space due to the lack of well-defined boundary conditions, leading to a loss of physical predictive power. Traditional methods introduce artificial boundary conditions or topological constraints but lack support from first principles. This paper proves that, by combining the fundamental postulates of quantum mechanics with the cosmological particle horizon (Rindler 1956), the physically relevant solutions of the Wheeler-DeWitt equation must belong to the space of bandlimited functions (Whittaker 1915). This selection is entirely based on the principle of observability and requires no presupposition of a transcendent external observer.

PHYSICAL POSTULATE AND REPRODUCING KERNEL HILBERT SPACE

A fundamental postulate of quantum mechanics is that a physical state is uniquely determined by the values of all possible observations on it. This means that if two states yield the same expectation values for all observables, they are the same state. Mathematically, this is equivalent to the state space being a reproducing kernel Hilbert space (Aronszajn 1950): there exists a positive definite kernel K such that every point evaluation is a continuous linear functional, and

$$\Psi[^{(3)}\mathcal{G}] = \langle \Psi, K(\cdot, ^{(3)}\mathcal{G}) \rangle. \tag{1}$$

For the Wheeler-DeWitt equation, the state functional Ψ is a functional of the three-geometry $^{(3)}\mathcal{G}$, and its dynamics is governed by the Hamiltonian constraint $\mathcal{H}\Psi = 0$ (DeWitt 1967). If Ψ belongs to a reproducing kernel Hilbert space, then the "global zero" condition holds (Falceto et al. 2013; Facchi et al. 2014): a state that yields zero expectation value for all observables must be the zero vector. Therefore, the reproducing kernel structure is an intrinsic requirement of quantum mechanics, not an artificial addition.

PARTICLE HORIZON: THE COSMOLOGICAL BOUNDARY OF OBSERVABILITY

Rindler (Rindler 1956), in his study of cosmology, pointed out that in models possessing a particle horizon (such as the big bang universe), any observer at a finite cosmic time t_0 can only receive information from within the horizon. The radius of the particle horizon is given by

$$l_{\text{horizon}}(t_0) = R(t_0) \int_0^{t_0} \frac{c\,dt}{R(t)}, \tag{2}$$

where $R(t)$ is the scale factor. Any event beyond the horizon has never entered the observer's past light cone and is therefore in principle unobservable. This means that, for a universe of finite age, the observable events constitute a discrete set whose distribution is determined by the causal structure of the universe. The cosmic wave function must be uniquely determined by its values on these discrete events and cannot depend on information from outside the horizon.

WHITTAKER'S CARDINAL FUNCTION: AN ANALYTIC FUNCTION UNIQUELY DETERMINED BY DISCRETE VALUES

Whittaker (Whittaker 1915), in his study of analytic function theory, proved a profound existence theorem: given a set of function values at equidistant points $a, a+w, a-w, a+2w, \ldots$, there exist infinitely many analytic functions that take the same values at these points (he called them "cotabular"). However, there is one and only one function that simultaneously satisfies two conditions:

1. it has no singularities in the finite part of the complex plane;
2. when analyzed by Fourier analysis, it has no oscillatory components with period less than $2w$.

This uniquely existing function he called the cardinal function, and its expression is

$$C(x) = \sum_{r=-\infty}^{\infty} f(a+rw) \frac{\sin\left(\frac{\pi}{w}(x-a-rw)\right)}{\frac{\pi}{w}(x-a-rw)}. \quad (3)$$

The key point: this is not a sampling theorem, but an existence and uniqueness theorem. In the context of the particle horizon, the discrete interval w is implicitly determined by the causal structure of the universe, and the conditions “no singularities” and “no short-period oscillations” correspond to the requirement of well-behavedness of physical reality. Therefore, the cosmic wave function, uniquely determined by discrete observational values within the horizon, must be a Whittaker cardinal function, and consequently its Fourier spectrum has compact support—i.e., it is bandlimited.

EQUIVALENCE OF REPRODUCING KERNEL AND BANDLIMITED SPACES

Aronszajn (Aronszajn 1950) established a one-to-one correspondence between reproducing kernel Hilbert spaces and positive definite kernels. For the bandlimited function space B^2_σ (the Paley-Wiener space), its reproducing kernel is the sinc kernel:

$$\frac{\sin \sigma(t-s)}{\pi(t-s)}. \quad (4)$$

Vaibhav (Vaibhav 2018), while studying the bandlimited extrapolation problem, pointed out that this kernel can be expanded as a series of spherical Bessel functions:

$$\frac{\sin \sigma(t-s)}{\pi(t-s)} = \sum_{n=0}^{\infty} \bar{j}_n(t)\bar{j}_n(\sigma s), \quad (5)$$

where $\bar{j}_n$ are normalized spherical Bessel functions. This identity shows that the bandlimited function space is equivalent to the space spanned by spherical Bessel functions, and spherical Bessel functions are naturally related to the isotropy of the universe. Thus, the bandlimited requirement imposed by the particle horizon is mathematically equivalent to the reproducing kernel structure, and hence consistent with the quantum mechanical postulate.

SOLUTION SPACE SELECTION OF THE WHEELER-DEWITT EQUATION

Applying the above conclusions to the Wheeler-DeWitt equation, the physically admissible solutions $\Psi[^{(3)}\mathcal{G}]$ must simultaneously satisfy:

1. They are uniquely determined by discrete geometric data within the horizon (the existence constraint of the particle horizon);
2. They have no singularities in the finite complex plane (the requirement of well-behavedness of physical reality);
3. They belong to a reproducing kernel Hilbert space (a direct consequence of the quantum postulate).

These three conditions collectively force Ψ to lie in the bandlimited function space. In other words, the solution space of the Wheeler-DeWitt equation is naturally truncated to

$$\mathcal{H}_{\text{phys}} = \left\{ \Psi \in \mathcal{H}_{\text{WDW}} : \text{supp}\, \hat{\Psi} \subset [-\sigma, \sigma] \right\}, \quad (6)$$

where σ is determined by the scale of the particle horizon. This truncation does not rely on artificial boundary conditions or an external observer; it is a mathematical necessity imposed by the very observability of the universe.

CONCLUSION

This paper proves that, in cosmological models possessing a particle horizon, the physically relevant solutions of the Wheeler-DeWitt equation must belong to the space of bandlimited functions. This result unifies Whittaker's existence theorem, Rindler's cosmological horizon concept, and Aronszajn's theory of reproducing kernels, demonstrating how the principle of observability mathematically constrains the solution space of quantum gravity equations. The proof requires no transcendent “God's eye” perspective; it is entirely based on causality and the fundamental postulates of quantum mechanics. Future work will explore the impact of this truncation on the specific form of the cosmic wave function and its compatibility with other boundary conditions in quantum cosmology.

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