Dual–Projection Informational Structures: A ZFC–Internal Extension of Informational Axiomatics

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Abstract

We introduce a ZFC–internal axiom system for dual–projection informational structures. The starting point is the single–projection framework of *informational models*—tuples $(X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta)$ consisting of a finitely additive measure, an idempotent projection, and an idempotent symmetric relation satisfying three short axioms coupling measure, projection, and curvature.

A dual–projection informational structure is an expansion of such a model by a second idempotent map $\Pi_M : X \to M_0$ and an equivalence relation $\approx$ whose classes are the fibres of Π_M. The new map is interpreted as a *structural* or *mathematical* projection. Basic axioms require that Π_M is measurable, measure–preserving on its range, and that $\approx$ is exactly the partition induced by Π_M.

We develop the elementary theory of dual–projection structures: we construct a canonical pushforward measure on the structural quotient, identify M_0 with the quotient $X/\approx$, and compare the physical equivalence relation induced by Π_R with the structural equivalence induced by Π_M. We prove that every informational model admits a canonical dual–projection expansion, so the new axioms are relatively consistent with the original system (assuming ZFC is). Finite and countable examples illustrate how a single set X can simultaneously support a time–asymmetric physical projection and a timeless structural projection.

Conceptually, dual–projection informational structures provide a discrete, ZFC–internal counterpart of dual–projection ontologies in which the "physical world" and the "mathematical world" are two projections of one informational ground. They are intended as the mathematical backbone for continuum formulations based on Informational Projection Theory and dual–projection informational ontology.

1 Introduction

This paper develops a minimal axiom system for *dual–projection* informational structures. It is designed as a direct extension of the single–projection framework of *informational models*, introduced in earlier work as tuples

$$(X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta)$$

satisfying three ZFC–internal axioms that tie together a finitely additive measure, an idempotent projection, and a symmetric idempotent relation.

Informational models were introduced to isolate, in a purely set–theoretic form, the interaction between:

- a finitely additive measure space $(X, \mathcal{A}, \mu)$;
- an idempotent retraction $\Pi_R : X \to R \subseteq X$ (a "physical" or "open" projection);
- an idempotent reflexive symmetric relation $G \subseteq X \times X$ (a curvature–like closure relation).

Three short axioms—duality/projection, closed curvature, and a conservation/coupling identity—were shown to admit finite and countable models, and to have nontrivial structural consequences.

Independently, philosophical work on informational ontology proposed a *dual–projection* picture: a single informational manifold admits two distinct projections. One projection, interpreted physically, generates a time–bound, entropic history by irreversibly hiding information. The other projection, interpreted mathematically, identifies configurations that share the same abstract structure and yields a timeless space of structures. A *projection symmetry* principle then characterises laws of nature as patterns invariant under both projections.

The aim of the present paper is to formulate the dual–projection idea at the same discrete, ZFC–internal level as the original informational axioms. We work entirely inside ZFC set theory; all objects are ordinary sets, functions, and relations.

We proceed as follows. Section 2 collects basic definitions and notation. Section 3 recalls the definition and axioms of informational models. Section 4 introduces dual–projection informational structures and states the new axioms. In Section 5 we derive structural properties of dual projections: quotient representations, pushforward measures, and equivalence relations. Section 6 establishes relative consistency by constructing dual–projection expansions of arbitrary informational models, as well as explicit finite and countable examples. Section 7 presents several concrete examples. Section 8 summarises and points towards continuum versions and physical applications.

Throughout we work inside ZFC. All sets and functions are assumed to be elements of the background universe.

2 Preliminaries

We fix some notation and basic definitions. Nothing here is new.

Definition 2.1 (Algebra of sets)**.** *Let X be nonempty. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an* algebra *if:*

- $\emptyset, X \in \mathcal{A}$*;*
- *whenever $B_1, B_2 \in \mathcal{A}$, then $B_1 \cup B_2 \in \mathcal{A}$ and $X \setminus B_1 \in \mathcal{A}$.*

Definition 2.2 (Finitely additive measure)**.** *Let $\mathcal{A}$ be an algebra on X. A map $\mu : \mathcal{A} \to [0, \infty)$ is a* finitely additive measure *if $\mu(\emptyset) = 0$ and $\mu(B_1 \sqcup B_2) = \mu(B_1) + \mu(B_2)$ whenever $B_1, B_2 \in \mathcal{A}$ are disjoint.*

Definition 2.3 (Binary relations and composition)**.** *A* binary relation *H on a set X is a subset of $X \times X$. The composition $H \circ K$ of relations $H, K \subseteq X \times X$ is*

$$H \circ K := \{(x, z) : \exists y \in X \text{ with } (x, y) \in H,\ (y, z) \in K\}.$$

The diagonal *is $\Delta_X := \{(x, x) : x \in X\}$. A relation H is* idempotent *if $H \circ H = H$.*

Definition 2.4 (Equivalence relation)**.** *A relation $\approx \subseteq X \times X$ is an* equivalence relation *if it is reflexive, symmetric, and transitive. For $x \in X$ we write $[x]_\approx := \{y \in X : y \approx x\}$ for its equivalence class.*

Definition 2.5 (Retraction)**.** *Let $R \subseteq X$. A map $\Pi_R : X \to R$ is a* retraction *if $\Pi_R|_R = \mathrm{id}_R$. It is* idempotent *if $\Pi_R \circ \Pi_R = \Pi_R$.*

Definition 2.6 (Product algebra and measure)**.** *Let $(X, \mathcal{A}, \mu)$ be a finitely additive measure space. The* product algebra *$\mathcal{A} \otimes \mathcal{A}$ is the algebra on $X \times X$ generated by rectangles $B_1 \times B_2$ with $B_i \in \mathcal{A}$. The* product measure *$\mu^{\otimes 2}$ on $\mathcal{A} \otimes \mathcal{A}$ is determined by $\mu^{\otimes 2}(B_1 \times B_2) := \mu(B_1)\mu(B_2)$ and finite additivity. When no confusion is likely, we use μ also for $\mu^{\otimes 2}$.*

3 Single–projection informational models

We recall the notion of an informational model and its axioms.

Definition 3.1 (Informational model)**.** *An* informational model *is a tuple*

$$M = (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta)$$

where:

- *X is a nonempty set and $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra;*
- *$\mu : \mathcal{A} \to [0, \infty)$ is a finitely additive measure;*
- *$R, I \in \mathcal{A}$ are disjoint and $R \cup I = X$;*
- *$\Pi_R : X \to R$ is a map;*
- *$G \subseteq X \times X$ lies in $\mathcal{A} \otimes \mathcal{A}$;*
- *$E_0 \in (0, \infty)$ and $\eta \in [0, 1]$ are scalars;*
- *the following axioms hold.*

Axiom I (duality and projection). $R \cap I = \emptyset$ and $R \cup I = X$. The map Π_R is idempotent and satisfies $\Pi_R|_R = \mathrm{id}_R$. If $B \in \mathcal{A} \cap \mathcal{P}(R)$, then $\Pi_R^{-1}(B) \in \mathcal{A}$.

Axiom II (closed curvature). The relation G is reflexive, symmetric, and idempotent: $G \circ G = G$.

Axiom III (conservation and coupling). The measure obeys $\mu(R) + \mu(I) = E_0 > 0$. If $B \in \mathcal{A} \cap \mathcal{P}(R)$, then $\mu(\Pi_R^{-1}(B)) = \mu(B)$. For all $B \in \mathcal{A} \cap \mathcal{P}(R)$,

$$\mu((B \times X) \cap G) = \mu(B) + \eta\, \mu((\Pi_R^{-1}(B) \times X) \cap G),$$

where μ on the right–hand side denotes the product measure on $\mathcal{A} \otimes \mathcal{A}$.

Lemma 3.2. *Let M be an informational model. Then:*
(a) *If $\Pi_R(x) = x$, then $x \in R$.*
(b) *For all $x \in X$, $\Pi_R(\Pi_R(x)) = \Pi_R(x)$.*

Proof. (a) If $\Pi_R(x) = x$, then x lies in the codomain of Π_R, which is R. (b) is exactly idempotence, part of Axiom I. □

4 Dual–projection informational structures

We now extend informational models by adding a second projection and an associated equivalence relation. Informally, the original map Π_R is regarded as a *physical* projection, while the new map Π_M captures *structural* or *mathematical* equivalence.

4.1 Definition and axioms

Definition 4.1 (Dual–projection informational structure)**.** *A* dual–projection informational structure *is a tuple*

$$\mathbf{M} = (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta; M_0, \Pi_M, \approx)$$

such that:
(D1) *The reduct $M := (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta)$ is an informational model in the sense of Definition 3.1.*
(D2) *$M_0 \subseteq X$ is nonempty and belongs to $\mathcal{A}$.*
(D3) *$\Pi_M : X \to M_0$ is an idempotent retraction: it satisfies $\Pi_M|_{M_0} = \mathrm{id}_{M_0}$ and $\Pi_M \circ \Pi_M = \Pi_M$. Moreover, whenever $B \in \mathcal{A} \cap \mathcal{P}(M_0)$, the preimage $\Pi_M^{-1}(B)$ lies in $\mathcal{A}$ and*

$$\mu(\Pi_M^{-1}(B)) = \mu(B).$$

(D4) *$\approx \subseteq X \times X$ is an equivalence relation.*
(D5) *For all $x, y \in X$,*

$$x \approx y \iff \Pi_M(x) = \Pi_M(y).$$

That is, the fibres of Π_M are exactly the $\approx$-classes.

We call Π_R the physical projection *and Π_M the* structural *(or* mathematical*) projection, and M_0 the* structural range.

Thus the new data $(M_0, \Pi_M, \approx)$ form a standard quotient picture: M_0 is a chosen set of representatives, Π_M picks a representative for each $x \in X$, and $\approx$ records when two points share the same representative.

Definition 4.1 does not impose any a priori interaction between Π_R and Π_M. The two projections are only linked indirectly through the shared underlying set and measure. In this paper we restrict ourselves to this minimal set of axioms; stronger bridge axioms can be added in future work if desired.

Remark 4.2. *One could equivalently take Π_M as primitive and define*

$$x \approx y \quad \text{iff} \quad \Pi_M(x) = \Pi_M(y),$$

dropping (D4) as an axiom. We keep $\approx$ explicit because in philosophical applications it is often convenient to speak of structural equivalence independently of the choice of representatives M_0.

4.2 Structural equivalence and quotients

We collect the basic consequences of the axioms for Π_M and $\approx$.

Lemma 4.3 (Equivalence classes as fibres)**.** *Let **M** be a dual–projection structure. For each $x \in X$,*

$$[x]_\approx = \Pi_M^{-1}(\Pi_M(x)).$$

In particular, the sets $[x]_\approx$ partition X.

Proof. By Axiom (D5) we have, for any $y \in X$,

$$y \in [x]_\approx \iff y \approx x \iff \Pi_M(y) = \Pi_M(x) \iff y \in \Pi_M^{-1}(\Pi_M(x)).$$

Thus the two sets coincide. Since $\approx$ is an equivalence relation, its classes form a partition. $\square$

Proposition 4.4 (Structural quotient)**.** *Let **M** be a dual–projection structure. Then the map $\Phi : M_0 \to X/\approx$ given by*

$$\Phi(m) := [m]_\approx$$

is a bijection. Its inverse $\Psi : X/\approx \to M_0$ sends $[x]_\approx \mapsto \Pi_M(x)$.

Proof. First note that if $m \in M_0$, then $\Pi_M(m) = m$ by Axiom (D3). Hence, by Lemma 4.3, $[m]_\approx = \Pi_M^{-1}(m) \neq \emptyset$.

Injectivity of Φ: if $\Phi(m_1) = \Phi(m_2)$, then $[m_1]_\approx = [m_2]_\approx$, so $m_1 \approx m_2$. By Axiom (D5), $\Pi_M(m_1) = \Pi_M(m_2)$, i.e. $m_1 = m_2$.

Surjectivity: for any class $[x]_\approx \in X/\approx$, set $m := \Pi_M(x) \in M_0$. Then by Lemma 4.3, $[x]_\approx = [m]_\approx = \Phi(m)$.

The proposed inverse Ψ is well–defined: if $[x]_\approx = [y]_\approx$, then $x \approx y$, hence $\Pi_M(x) = \Pi_M(y)$ by Axiom (D5); so Ψ does not depend on the chosen representative. Finally, $\Psi \circ \Phi = \mathrm{id}_{M_0}$ and $\Phi \circ \Psi = \mathrm{id}_{X/\approx}$ follow directly from the definitions. $\square$

The upshot is that, up to canonical bijection, one may view M_0 simply as the quotient $X/\approx$. The explicit inclusion of both is purely a matter of convenience.

4.3 Structural measure

The next proposition shows that the measure μ can be pushed forward along Π_M to the structural range M_0.

Proposition 4.5 (Pushforward structural measure)**.** *Let* $\mathbf{M}$ *be a dual–projection structure. Define*

$$\mathcal{B}_M := \{B \subseteq M_0 : \Pi_M^{-1}(B) \in \mathcal{A}\}.$$

Then:

(a) *$\mathcal{B}_M$ is an algebra of subsets of M_0;*
(b) *the map $\nu : \mathcal{B}_M \to [0, \infty)$ given by $\nu(B) := \mu(\Pi_M^{-1}(B))$ is a finitely additive measure;*
(c) *on the subalgebra $\mathcal{A}_0 := \mathcal{A} \cap \mathcal{P}(M_0) \subseteq \mathcal{B}_M$ we have $\nu(B) = \mu(B)$ for all $B \in \mathcal{A}_0$.*

Proof. (a) Since $\Pi_M^{-1}(\emptyset) = \emptyset \in \mathcal{A}$ and $\Pi_M^{-1}(M_0) = X \in \mathcal{A}$, we have $\emptyset, M_0 \in \mathcal{B}_M$. If $B_1, B_2 \in \mathcal{B}_M$, then $\Pi_M^{-1}(B_1), \Pi_M^{-1}(B_2) \in \mathcal{A}$, and

$$\Pi_M^{-1}(B_1 \cup B_2) = \Pi_M^{-1}(B_1) \cup \Pi_M^{-1}(B_2) \in \mathcal{A},$$

so $B_1 \cup B_2 \in \mathcal{B}_M$. Similarly,

$$\Pi_M^{-1}(M_0 \setminus B_1) = X \setminus \Pi_M^{-1}(B_1) \in \mathcal{A},$$

so $M_0 \setminus B_1 \in \mathcal{B}_M$. Thus $\mathcal{B}_M$ is an algebra.

(b) Let $B \in \mathcal{B}_M$. Define $\nu(B) := \mu(\Pi_M^{-1}(B))$. Then $\nu(\emptyset) = \mu(\emptyset) = 0$. If $B_1, B_2 \in \mathcal{B}_M$ are disjoint, then

$$\Pi_M^{-1}(B_1 \sqcup B_2) = \Pi_M^{-1}(B_1) \sqcup \Pi_M^{-1}(B_2),$$

so finite additivity of μ gives

$$\nu(B_1 \sqcup B_2) = \mu(\Pi_M^{-1}(B_1 \sqcup B_2)) = \mu(\Pi_M^{-1}(B_1)) + \mu(\Pi_M^{-1}(B_2)) = \nu(B_1) + \nu(B_2).$$

(c) If $B \in \mathcal{A}_0 = \mathcal{A} \cap \mathcal{P}(M_0)$, then by Axiom (D3) we have $\Pi_M^{-1}(B) \in \mathcal{A}$ and $\mu(\Pi_M^{-1}(B)) = \mu(B)$. By definition of ν, $\nu(B) = \mu(\Pi_M^{-1}(B)) = \mu(B)$. □

Thus every dual–projection structure carries, in addition to the original measure μ on $(X, \mathcal{A})$, a natural "structural measure" ν on the quotient M_0.

5 Consequences for the physical projection

The new data $(M_0, \Pi_M, \approx)$ do not alter the original axioms for Π_R, but they allow us to compare the quotients induced by the two projections.

5.1 Physical equivalence relation

Definition 5.1 (Physical equivalence)**.** *Let* $\mathbf{M}$ *be a dual–projection structure. Define a relation $\sim_R$ on X by*

$$x \sim_R y \iff \Pi_R(x) = \Pi_R(y).$$

We call $\sim_R$ the physical equivalence relation *induced by Π_R.*

Lemma 5.2. *The relation $\sim_R$ is an equivalence relation on X. For each $x \in X$, its $\sim_R$–class is $[x]_{\sim_R} = \Pi_R^{-1}(\Pi_R(x))$.*

Proof. Reflexivity follows from idempotence: $\Pi_R(x) = \Pi_R(\Pi_R(x))$, so $x \sim_R x$. Symmetry and transitivity are immediate from equality of $\Pi_R(x)$. The description of the classes is analogous to Lemma 4.3. □

Restricting $\sim_R$ to the open sector R simplifies the picture.

Lemma 5.3 (Restriction to R)**.** *Let* $\mathbf{M}$ *be a dual–projection structure and* $x, y \in R$*. Then*

$$x \sim_R y \iff x = y.$$

In particular, every $\sim_R$*–class contained in* R *is a singleton, and the quotient* $R/\sim_R$ *can be canonically identified with* R *itself.*

Proof. If $x, y \in R$ and $\Pi_R(x) = \Pi_R(y)$, then $\Pi_R(x) = x$ and $\Pi_R(y) = y$ by Lemma 3.2(a), so $x = y$. The converse is trivial. The remaining statements follow. □

Thus the interesting physical quotient structure occurs at the level of all of X, not within R. Inside R the projection Π_R acts as the identity.

Remark 5.4. *In many intended applications, one interprets the full* $\sim_R$*–classes* $\Pi_R^{-1}(r)$ *as sets of informational configurations that realise the same open (physical) state* $r \in R$*. The measure* μ *of such classes then plays the role of an entropy–like quantity.*

5.2 Canonical map on the open sector

There is no axiom directly relating $\approx$ and $\sim_R$. However, using Lemma 5.3 we can still define a canonical map from the physical quotient restricted to R into the structural range.

Proposition 5.5 (Canonical map on R)**.** *Let* $\mathbf{M}$ *be a dual–projection structure. Define* $\Theta : R/\sim_R \to M_0$ *by*

$$\Theta([x]_{\sim_R}) := \Pi_M(x), \qquad x \in R.$$

Then Θ *is well–defined. Under the canonical identification* $R/\sim_R \cong R$ *from Lemma 5.3,* Θ *is just the restriction of* Π_M *to* R*.*

Proof. If $[x]_{\sim_R} = [y]_{\sim_R}$ with $x, y \in R$, then $\Pi_R(x) = \Pi_R(y)$, hence $x = y$ by Lemma 5.3. Thus the class $[x]_{\sim_R}$ contains a unique element, and $\Pi_M(x)$ does not depend on the choice of representative. The last statement is immediate. □

In particular, if one is interested only in the open sector R, the dual projection Π_M does not interact with Π_R in any nontrivial way at the purely set–theoretic level; their interaction is mediated through measure and additional structures (e.g. dynamics) that are not built into the present axioms.

6 Dual–projection models and relative consistency

We now establish that the dual–projection axioms are relatively consistent with the axioms of informational models, by constructing explicit expansions. The key point is that every informational model admits a *canonical* dual–projection expansion, and that there also exist expansions with $\Pi_M \neq \Pi_R$.

6.1 Canonical trivial expansion

Theorem 6.1 (Canonical dual–projection expansion)**.** *Let*

$$M = (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta)$$

be an informational model. Define

$$M_0 := R, \qquad \Pi_M := \Pi_R, \qquad x \approx y \iff \Pi_R(x) = \Pi_R(y).$$

Then

$$\mathbf{M} := (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta; M_0, \Pi_M, \approx)$$

is a dual–projection informational structure.

Proof. Axioms (D1) and (D2) hold by assumption and the fact that $R \in \mathcal{A}$. For Axiom (D3), $\Pi_M = \Pi_R$ is idempotent and restricts to the identity on $R = M_0$. Measurability and measure–preservation of preimages for subsets of R are exactly the measurability and invariance clauses in Axiom III of informational models.

Define $\approx$ by equality of Π_R–values. Then $\approx$ is an equivalence relation (it is the kernel of a map), so Axiom (D4) holds. Finally, by construction,

$$x \approx y \iff \Pi_R(x) = \Pi_R(y) \iff \Pi_M(x) = \Pi_M(y),$$

so Axiom (D5) holds. Thus $\mathbf{M}$ is a dual–projection structure. □

Corollary 6.2 (Relative consistency)**.** *If ZFC is consistent and the axioms of informational models are consistent with ZFC, then the axioms of dual–projection informational structures are also consistent with ZFC.*

Proof. Any model of the informational axioms expands to a model of the dual–projection axioms by Theorem 6.1. □

The expansion in Theorem 6.1 is *trivial* in the sense that the structural projection coincides with the physical projection. Nevertheless it suffices for relative consistency.

6.2 Finite models with nontrivial dual projection

We now exhibit dual–projection structures in which the two projections differ. We start from finite informational models.

Theorem 6.3 (Finite dual–projection model with $\Pi_M \neq \Pi_R$)**.** *There exists a finite dual–projection informational structure in which $\Pi_M \neq \Pi_R$.*

Proof. Let $M = (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta)$ be a finite informational model as constructed in the original axiomatic work: X is finite, $\mathcal{A} = \mathcal{P}(X)$, and $R \subseteq X$ is nonempty.

Define $M_0 := X$ and let $\Pi_M : X \to M_0 = X$ be the identity map: $\Pi_M(x) := x$. Then Π_M is an idempotent retraction and for any $B \subseteq M_0$ we have $\Pi_M^{-1}(B) = B \in \mathcal{A}$. Moreover, $\mu(\Pi_M^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{A}$, so Axiom (D3) is satisfied. Define $\approx$ to be equality:

$$x \approx y \iff x = y.$$

Then $\approx$ is an equivalence relation and clearly satisfies Axiom (D5), since $\Pi_M(x) = \Pi_M(y)$ iff $x = y$.

Thus $(M_0, \Pi_M, \approx)$ satisfies Axioms (D2)–(D5), and together with the original informational structure we obtain a finite dual–projection informational structure. Since Π_M is the identity whereas Π_R is not (it strictly retracts onto $R \subsetneq X$), we have $\Pi_M \neq \Pi_R$. □

In this simple example, the structural projection sees all informational distinctions, while the physical projection collapses closed information onto the open set R. More elaborate finite examples can be obtained by choosing intermediate structural ranges M_0 and appropriate partitions of X; we do not pursue this in detail here, as the construction is routine.

6.3 Countable models

Combining Theorem 6.1 with known constructions of countable informational models yields countable dual–projection models.

Theorem 6.4 (Countable dual–projection models)**.** *There exists a dual–projection informational structure with X countably infinite.*

Proof. The original axiomatic work constructs an informational model

$$M = (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta)$$

with X countably infinite. Applying Theorem 6.1 to this M yields a dual–projection structure on the same underlying set X. □

Thus the dual–projection axioms have both finite and countable models.

7 Examples

We collect several examples and patterns that can be captured in dual–projection informational structures. In each case the focus is on the additional data $(M_0, \Pi_M, \approx)$; the underlying informational model may be obtained either by the finite and countable constructions of Section 6 or by the general existence results of the single–projection axioms.

7.1 Finite open/closed pattern with structural types

Let X be a finite set of "microconfigurations" and suppose it is partitioned into finitely many nonempty blocks $(C_j)_{j \in J}$. Informally, each block C_j collects informational states that are equivalent for the purposes of a chosen physical description.

Choose one representative $r_j \in C_j$ from each block and set

$$R := \{r_j : j \in J\}, \qquad I := X \setminus R.$$

Let $\mathcal{A} := \mathcal{P}(X)$, and define $\Pi_R : X \to R$ by $\Pi_R(x) := r_j$ whenever $x \in C_j$. Choose weights $w_j \geq 0$ with $\sum_j w_j > 0$ and define $\mu : \mathcal{A} \to [0, \infty)$ by

$$\mu(B) := \sum_{j \in J : r_j \in B} w_j.$$

Let $G \subseteq X \times X$ be the union of the diagonal blocks $C_j \times C_j$, as in the finite models constructed in Section 6. With suitable choices of E_0 and η, this yields a finite informational model.

The canonical dual–projection expansion from Theorem 6.1 sets

$$M_0 := R, \qquad \Pi_M := \Pi_R, \qquad x \approx y \iff \Pi_R(x) = \Pi_R(y).$$

In this interpretation:
- each block C_j is the set of microstates that share one open (physical) state r_j;
- the physical projection Π_R selects a single open representative from each block;
- the structural projection Π_M coincides with Π_R and identifies configurations differing only by closed information.

Although here Π_R and Π_M agree, the example is useful as a minimal template for more interesting expansions.

7.2 Abstract equivalence/bridge pattern

Let X be any nonempty set equipped with an equivalence relation $\approx$. Think of $\approx$ as expressing "having the same structure" in some chosen sense. Choose one representative from each equivalence class to form a set $M_0 \subseteq X$, and define $\Pi_M : X \to M_0$ by mapping each $x \in X$ to the unique $m \in M_0$ with $x \approx m$. Then Π_M is an idempotent retraction and, by construction, its fibres are exactly the $\approx$–classes, so $(M_0, \Pi_M, \approx)$ satisfies Axioms (D2)–(D5).

If one can equip X with an algebra $\mathcal{A}$, a finitely additive measure μ, a partition $X = R \sqcup I$, a retraction $\Pi_R : X \to R$, and a curvature relation $G \subseteq X \times X$ satisfying the informational axioms, then

$$\mathbf{M} = (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta; M_0, \Pi_M, \approx)$$

is a dual–projection informational structure. The structural measure ν from Proposition 4.5 then lives on M_0, which may be identified with the quotient $X/\approx$, whereas the physical measure μ lives on $(X, \mathcal{A})$, with Π_R encoding which informational differences matter physically.

7.3 Graph isomorphism as structural equivalence

We next give a concrete finite example in which the physical and structural projections are genuinely different. Fix $n \in \mathbb{N}$, and let X be the finite set of all simple graphs with vertex set $\{1, \ldots, n\}$. Each graph $x \in X$ may be represented by its adjacency matrix.

There are two natural notions of equivalence on X.

- Two graphs are *physically equivalent* if they have the same degree sequence (up to permutation of vertices).
- Two graphs are *structurally equivalent* if they are isomorphic as unlabeled graphs.

Underlying informational model. Let $\mathcal{A} := \mathcal{P}(X)$. For each degree sequence d that arises among graphs on n vertices, let

$$C_d := \{x \in X : \text{the degree sequence of } x \text{ is } d\}.$$

The sets C_d form a finite partition of X. Choose one labelled graph $r_d \in C_d$ as a "degree–canonical" representative and set

$$R := \{r_d : d\}, \qquad I := X \setminus R.$$

Define $\Pi_R : X \to R$ by $\Pi_R(x) := r_d$ whenever $x \in C_d$. As in the finite model construction of Section 6, choose weights $w_d \geq 0$ with $\sum_d w_d > 0$ and define $\mu(B) := \sum_{d:\, r_d \in B} w_d$. Let

$$G := \bigcup_d (C_d \times C_d) \subseteq X \times X.$$

With suitable choices of E_0 and η this yields a finite informational model: the blocks C_d play the role of curvature–classes and Π_R is the physical projection that remembers only the degree sequence.

Structural projection. Now consider graph isomorphism. Partition X into isomorphism classes $(D_i)_{i \in I}$, where each D_i is a set of graphs that are mutually isomorphic as unlabeled graphs. For each $i \in I$, choose one canonical labelled representative $m_i \in D_i$ (for instance, the graph whose adjacency matrix is lexicographically minimal) and set

$$M_0 := \{m_i : i \in I\} \subseteq X.$$

Define $\Pi_M : X \to M_0$ by letting $\Pi_M(x)$ be the unique m_i with $x \in D_i$, and define an equivalence relation $\approx$ on X by

$$x \approx y \quad \Longleftrightarrow \quad \Pi_M(x) = \Pi_M(y),$$

i.e. x and y are isomorphic graphs. Then:

- Π_M is an idempotent retraction onto M_0;
- $\approx$ is an equivalence relation whose classes are exactly the fibres of Π_M;
- since $\mathcal{A} = \mathcal{P}(X)$, the preimage $\Pi_M^{-1}(B)$ lies in $\mathcal{A}$ for every $B \subseteq M_0$.

To satisfy the measure–preservation clause in Axiom (D3), we can choose μ so that it is supported only on a subset $S \subseteq R \cap M_0$ of graphs that are simultaneously degree–canonical and structural representatives, assigning weights $w_s \geq 0$ to $s \in S$ and $\mu(B) = \sum_{s \in B \cap S} w_s$. Then for any $B \subseteq M_0$, the sets B and $\Pi_M^{-1}(B)$ differ only by graphs outside S, which all have measure zero, so $\mu(\Pi_M^{-1}(B)) = \mu(B)$. Similarly, for $B \subseteq R$, $\Pi_R^{-1}(B)$ and B differ only by graphs outside S, so $\mu(\Pi_R^{-1}(B)) = \mu(B)$. Thus Axiom (D3) holds simultaneously for Π_R and Π_M.

Hence $(M_0, \Pi_M, \approx)$ satisfies Axioms (D2)–(D5), and together with the underlying informational model we obtain a finite dual–projection structure

$$\mathbf{M} = (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta; M_0, \Pi_M, \approx).$$

Interpretation. In this example:

- the physical projection Π_R coarse–grains graphs by degree sequence, so many nonisomorphic graphs are physically indistinguishable;
- the structural projection Π_M identifies graphs up to isomorphism, so that M_0 may be identified with the finite set of unlabeled graphs on n vertices;
- every isomorphism class D_i is contained in one degree block C_d, so structural equivalence refines physical equivalence.

Thus the same underlying set X supports a coarse physical classification and a finer structural classification.

7.4 Rays in a countable Hilbert space

We now sketch a countable example in which the structural projection recovers the usual identification of quantum states that differ only by a global phase or overall scalar factor.

Let H be a separable complex Hilbert space with a fixed orthonormal basis $(e_k)_{k \in \mathbb{N}}$. Let X be the set of all nonzero vectors

$$x = \sum_{k=0}^{N_x} c_k e_k$$

with finite support and coefficients $c_k \in \mathbb{Q}(i)$ (Gaussian rationals). Then X is countable. Let $\mathcal{A} := \mathcal{P}(X)$. Equip X with a finitely additive measure $\mu : \mathcal{A} \to [0, \infty)$ and a physical projection Π_R, curvature relation G, etc., so that $(X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta)$ is an informational model. (For instance, one may adapt the countable block constructions of Section 6 by partitioning X according to coarse measurement outcomes.)

Structural equivalence as projective equivalence. Define an equivalence relation $\approx$ on X by

$$x \approx y \quad \Longleftrightarrow \quad \exists \lambda \in \mathbb{Q}(i) \setminus \{0\} \text{ such that } y = \lambda x.$$

Thus two rational vectors are structurally equivalent if they differ only by a nonzero scalar multiple (with rational complex scalar). Each equivalence class is countably infinite.

For each nonzero $x \in X$, let $k(x)$ be the least index k with coefficient $c_k \neq 0$. Define a canonical representative $\widetilde{x}$ of the $\approx$–class of x by rescaling so that the first nonzero coefficient equals 1:

$$\widetilde{x} := \frac{1}{c_{k(x)}} x.$$

Since $c_{k(x)} \in \mathbb{Q}(i) \setminus \{0\}$, the vector $\widetilde{x}$ again belongs to X. Let

$$M_0 := \{\widetilde{x} : x \in X\} \subseteq X.$$

Define $\Pi_M : X \to M_0$ by $\Pi_M(x) := \widetilde{x}$.

Verification of the axioms. By construction:

- Π_M is idempotent: if $x \in M_0$, then its first nonzero coefficient is already 1, so $\Pi_M(x) = x$;
- Π_M is a retraction onto M_0: it restricts to the identity on M_0;
- $\approx$ is an equivalence relation and $x \approx y$ iff $\Pi_M(x) = \Pi_M(y)$;
- since $\mathcal{A} = \mathcal{P}(X)$, the preimage $\Pi_M^{-1}(B)$ belongs to $\mathcal{A}$ for every $B \subseteq M_0$.

To ensure Axiom (D3), we may choose μ to be supported only on M_0, assigning nonnegative weights w_m to $m \in M_0$ and defining $\mu(B) := \sum_{m \in B \cap M_0} w_m$ for all $B \subseteq X$. Then any $x \notin M_0$ has measure zero, and for every $B \subseteq M_0$ we have $\Pi_M^{-1}(B) = B \cup N_B$ where $N_B \subseteq X \setminus M_0$ consists of nonrepresentatives. Hence

$$\mu(\Pi_M^{-1}(B)) = \mu(B \cup N_B) = \mu(B),$$

and the measure–preservation clause of Axiom (D3) holds. The requirements of the informational axioms for Π_R can be met by taking $R \subseteq M_0$ and defining Π_R and G using the same block construction as in Section 6.

Thus $(M_0, \Pi_M, \approx)$ satisfies Axioms (D2)–(D5), and together with the chosen informational structure on X we obtain a countable dual–projection model.

Interpretation. Here the structural quotient M_0 may be identified with a countable subset of the projective Hilbert space PH (the set of rays in H). The equivalence relation $\approx$ encodes the usual identification of quantum states that differ only by a global scalar factor; the structural projection Π_M takes a concrete rational superposition and returns a canonical representative of its projective class.

The physical projection Π_R can be chosen to represent coarse macroscopic information (for instance, pointer eigenstates of some chosen observable), so that Π_R forgets most of the Hilbert–space detail while Π_M retains the full projective structure. The two projections thus formalise, in a discrete setting, the separation between physical measurement outcomes and the abstract space of quantum states.

7.5 Toy model of an informational black–hole horizon

As a final example, we sketch a finite dual–projection structure intended as a toy model for black–hole microstates and horizon entropy. The idea is to distinguish:

- a *physical* projection that only sees a subset of macroscopic parameters (for instance, mass);
- a *structural* projection that distinguishes fuller sets of parameters (mass, charge, angular momentum, and possibly discrete hair).

Microstates and macroscopic parameters. Fix a finite set S of macroscopic parameter tuples, for example

$$S \subseteq \mathbb{Q}^4, \qquad s = (M, J, Q, N),$$

where M is mass, J angular momentum, Q charge, and N a discrete label (e.g. a topological charge or species index). For each $s \in S$, choose a finite nonempty set F_s of "microstates" that realise the macroscopic parameters s, and let

$$X := \bigcup_{s \in S} F_s.$$

Thus each $x \in X$ carries two kinds of information: a macroscopic label $s(x) \in S$ and a finer "internal" label distinguishing microstates with the same macroscopic parameters.

Underlying informational model. Now forget the fine labels and group microstates by *mass* alone. Let $\{M_j : j \in J\}$ be the distinct mass values appearing in S, and let

$$C_j := \{x \in X : s(x) = (M_j, J, Q, N) \text{ for some } (J, Q, N)\}.$$

The sets C_j form a finite partition of X into "mass levels". Choose for each $j \in J$ a distinguished microstate $r_j \in C_j$, and set

$$R := \{r_j : j \in J\}, \qquad I := X \setminus R.$$

Let $\mathcal{A} := \mathcal{P}(X)$, and define $\Pi_R : X \to R$ by $\Pi_R(x) := r_j$ whenever $x \in C_j$. Choose weights $w_j \geq 0$ with $\sum_j w_j > 0$ and define $\mu(B) := \sum_{j : r_j \in B} w_j$. Let

$$G := \bigcup_{j \in J} (C_j \times C_j) \subseteq X \times X.$$

Exactly as in the finite block model of Section 6, this yields an informational model. Intuitively, Π_R forgets all details behind the horizon except for the total mass M_j; the curvature relation G groups microstates within the same mass level.

Structural projection. Define an equivalence relation $\approx$ on X by declaring that

$$x \approx y \quad \Longleftrightarrow \quad s(x) = s(y),$$

i.e. x and y share the full macroscopic parameter tuple (M, J, Q, N). For each $s \in S$, choose one canonical microstate $m_s \in F_s$ and set

$$M_0 := \{m_s : s \in S\} \subseteq X.$$

Define $\Pi_M : X \to M_0$ by $\Pi_M(x) := m_{s(x)}$. Then:

- Π_M is an idempotent retraction onto M_0;
- $\approx$ is an equivalence relation and $x \approx y$ iff $\Pi_M(x) = \Pi_M(y)$;
- since $\mathcal{A} = \mathcal{P}(X)$, the preimage $\Pi_M^{-1}(B)$ is measurable for all $B \subseteq M_0$.

To satisfy Axiom (D3), we may choose μ so that it is supported only on the set $R \subseteq X$ of mass representatives, as in the underlying informational model, and at the same time choose each canonical m_s so that whenever a mass level C_j contains a state with parameters $s = (M_j, J, Q, N)$, the corresponding m_s is either r_j or a microstate in the same $\approx$–class as r_j. In this way, for any $B \subseteq M_0$ the sets B and $\Pi_M^{-1}(B)$ differ only by microstates of measure zero, and hence $\mu(\Pi_M^{-1}(B)) = \mu(B)$. The invariance condition for Π_R in Axiom III was already enforced by the original choice of μ.

Thus $(M_0, \Pi_M, \approx)$ satisfies Axioms (D2)–(D5), and we obtain a finite dual–projection structure

$$\mathbf{M} = (X, \mathcal{A}, \mu, R, I, \Pi_R, G, E_0, \eta; M_0, \Pi_M, \approx).$$

Interpretation. In this toy model:

- the physical projection Π_R sees only the mass M_j, as might be the case for an observer with access only to asymptotic gravitational data;
- the structural projection Π_M distinguishes configurations with the same mass but different (J, Q, N), i.e. different macroscopic charges or discrete hair;
- the equivalence classes of $\approx$ (the sets F_s) and the finer internal labels inside each F_s encode degeneracies of the horizon microstates that do not affect the coarse mass observable.

If $|F_s|$ is large for many s, then the logarithm of the total number of microstates with a given mass M_j provides a discrete analogue of Bekenstein–Hawking entropy in this setting. The dual–projection structure separates the coarse exterior description (mass levels) from the finer structural classification (full parameter tuples and microstate degeneracy) within a single ZFC–internal framework.

8 Conclusion and outlook

We have introduced *dual–projection informational structures* as ZFC–internal expansions of informational models by a second idempotent projection and an associated equivalence relation. The new data $(M_0, \Pi_M, \approx)$ are required to form a quotient–like structure, measurable with respect to the underlying algebra and measure, and to be measure–preserving on their range. From these axioms we obtained:

- a canonical identification of M_0 with the quotient $X/\approx$;
- a natural pushforward measure ν on the structural range;
- a comparison between the physical equivalence relation induced by Π_R and the structural equivalence induced by Π_M;
- explicit finite and countable models, and a canonical way to expand any informational model to a dual–projection one.

Mathematically, the theory developed here is intentionally modest. It shows that adding a second projection and a quotient structure on the same underlying set does not introduce inconsistencies, and that both projections can coexist with a common finitely additive measure and curvature relation. The framework is flexible: stronger bridge axioms relating Π_R and Π_M—for example, axioms expressing "projection symmetry" through joint invariants—can be added on top of the present system without changing its basic set–theoretic character.

Conceptually, dual–projection informational structures provide a discrete arena in which ideas from dual–projection ontology and Informational Projection Theory can be made precise. At the continuum level, one can replace the finite algebra $\mathcal{A}$ by a σ–algebra on a manifold, reinterpret G as a closed curvature form, and couple both projections to dynamical laws. The hope is that the simple, ZFC–internal formalism developed here will serve as a clean reference point for that more ambitious programme.

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