Variational Backbone and Regime Closures XIII:
Gauge Forces as Operator Instances and RG Readouts
Yang–Mills principal operator and asymptotic freedom as a scale-instance output

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Abstract

We place gauge forces as operator instances on a fixed variational backbone and locate logarithmic running (including asymptotic freedom) at the rail/scale-instance level. Within the density-fixed contract and principal-symbol invariance under admissible elimination, we state a canonical Yang–Mills principal-operator normal form and separate S0-type representative changes (principal-operator invariant) from RG readouts (scale-dependent outputs).

Two instance signatures.
$\mathbf{Inst}_{\mathrm{gauge}}$: operator-instance choice on the fixed backbone (Yang–Mills principal operator normal form).
$\mathbf{Inst}_{\mathrm{RG}}(\mu)$: scale/readout instance determining running as a *rail-level* output (a μ-dependent readout).

Protocol usage. This part uses the Representation Layer protocol template (Part IV): it relies on readout/retention (P1) and induced weights/updates (P2), and any additional closure/coupling slot (P3) is stated explicitly when used. Elimination/projector/Schur procedures are representation-level operations induced by the declared protocol and never modify the Structural Layer.

Self-contained review note. This manuscript is intended to be *self-contained* for peer review. Companion parts of the series are cited only for context and do not supply any result required for the arguments below. Whenever a series-level convention is used (equivalences, instance vocabulary, gates/rails), it is restated locally or declared explicitly.

Meta-language. Nonstandard series terms (reading, instance alias, gate, rail, principal operator lock, etc.) are defined in the *Series meta-language glossary* appended to this manuscript.

Scope. All claims are *sectorial/regime-level* unless explicitly stated: validity and breakdown are controlled by declared regime gates and stopping rules, and no global completeness or continuation is asserted beyond regime exits.

Series pipeline (structure at a glance). Primitive E and constitutional axioms $\rightarrow$ reading (R1/R2/R3) $\rightarrow$ elimination (Schur/dual-Schur) $\rightarrow$ representation instances $\rightarrow$ (optional) rails for validation. Companion parts of the series develop: (i) metric-channel regime templates, (ii) field-sector regime templates (KG$\rightarrow$Sch$\rightarrow$HJ), (iii) a time-instance module, and (iv) sectorial closure dossiers (classical/GR/QM/UV), each stated to be review-independent.

Companion parts. Other parts of *Variational Backbone and Regime Closures* may be consulted for background only (see the series map). They do not supply any result required for the arguments in the present manuscript. Dedicated validation rails (cosmology/lensing/data comparison) and optional phenomenology instances (e.g. PPN readouts) are deferred to follow-up dossiers. No step in this manuscript depends on an unpublished argument.

Series scope. All statements are sectorial/regime-level within the declared regime and admissibility gates, and are subject to explicit stopping rules where readout instances are used; no global continuation beyond regime exit is claimed.

Keywords: gauge theory; Yang–Mills; renormalization group; asymptotic freedom; operator instance; scale instance.

Contents

Scope note. References to elimination in this paper follow the series contract: elimination is a representation-level representative selection; Part VII states its admissibility (principal operator invariance).

1 Introduction and positioning

This paper is a module extension in the VBRC series. It fixes *where* gauge forces and running couplings live within the established layer separation (structure/instances/gates/rails). Parts I–XII fix the density-fixed backbone contract, the operational readings (R1/R2/R3), and the principal-operator invariance under admissible eliminations (Part VII). Here we add: (i) a gauge-sector *operator instance* $\mathbf{Inst}_{\text{gauge}}$ and (ii) a *scale (RG) instance* $\mathbf{Inst}_{\text{RG}}(\mu)$. The separation is strict: pre-instance structural invariants (principal symbol, characteristic cone, kernel charges) are fixed within the admissible regime, while scale-dependent couplings such as $g(\mu)$ are rail outputs by construction [1].

Scope. The Standard Model gauge group and matter representations are treated as representation-level modeling choices (not fixed as structural theorems here). Group selection, anomaly constraints, electroweak symmetry breaking, and precision fits are deferred to rail-oriented dossiers.

Example (instance only). For orientation only, one may instantiate $\mathbf{Inst}_{\text{gauge}}$ by the Standard Model group $SU(3) \times SU(2) \times U(1)$; no uniqueness is claimed.

2 Gauge operator instance

Definition 13.1 (Gauge operator instance $\mathbf{Inst}_{\text{gauge}}$)**.** A gauge operator instance consists of: (i) a compact Lie group G and a principal G-bundle (or an associated G-vector bundle), (ii) a connection A defining a covariant derivative $D := \nabla + A$, and (iii) the curvature $F := D^2$ (equivalently $F = dA + A \wedge A$ in a local trivialization). The instance specifies the representation of matter fields (if present) and fixes the treatment of gauge redundancy.

Remark 13.1 (Consistency with the representation-protocol template)*.* The RG-instance viewpoint is consistent with the Part IV representation-protocol template (AIS: Axiomatic Instance Selection; one admissible instance): a change of scale is a change of readout and normalization. In particular, the choice of base weight/normalization μ controls whether the coupling remains nontrivial ($\eta \neq 0$) or collapses to a trivial regime ($\eta = 0$), as in the σ-additive/probability-normalized versus large-mass regimes in the axiomatic analysis.

3 Yang–Mills principal operator as an admissible normal form

Theorem 13.1 (Yang–Mills principal operator as an admissible gauge kinetic normal form)**.** *Within the density-fixed contract of Part I (first-derivative locality and principal-operator control), and under admissible elimination in the sense of Part VII (order-*0 *modifications only), any gauge-sector representative whose derivative-dependent part is (i) gauge-invariant, (ii) quadratic in first derivatives, and (iii) does not modify the principal symbol class, admits a canonical normal-form representative whose leading gauge kinetic term is*

$$E_{\mathrm{YM}}[A] \;=\; \frac{1}{2g_0^2}\int_M \langle F, F\rangle \,\mathrm{dvol}_g,$$

*up to declared equivalences (boundary terms, constant rescalings, and admissible lower-order stabilizations). In particular, admissible eliminations may generate only order-*0 *corrections and do not alter the gauge-sector principal operator.*

Proof. Fix $\mathbf{Inst}_{\text{gauge}}$ and consider any local, gauge-invariant functional whose derivative-dependent part is quadratic and of first-derivative order in the connection data. Gauge invariance forces the derivative-dependent quantity to factor through the curvature F, since F is the unique covariant first-derivative tensor built from A (up to bundle/representation conventions). Quadraticity then yields a leading term of the form $\int \langle F, \mathcal{K}F\rangle \,\mathrm{dvol}_g$ with an order-0 bundle endomorphism $\mathcal{K}$. Within the declared admissible equivalences (Part VII), $\mathcal{K}$ may be absorbed into a constant rescaling of the coupling and an inner-product choice, leaving the canonical representative $\int \langle F, F\rangle$. Any additional gauge-invariant terms that are local and do not alter the principal symbol are necessarily order-0 (mass/potential-type, gauge-fixing conventions, or stabilizations) or boundary/topological terms, hence are S0-admissible and do not change the principal operator. □

Remark 13.2 (Product groups and multiple couplings (notation lock))*.* If the gauge group decomposes as a product of simple and abelian factors, e.g. $G \cong \prod_a G_a \times U(1)^r$, then the same principal-level argument yields a *sum* of Yang–Mills/Maxwell principal operators, one per factor, each with its own coupling g_a and its own chosen invariant inner product on $\mathfrak{g}_a$ (equivalently, a normalization choice for the invariant inner product on that factor). In this part we suppress factor indices and write a single g and a single invariant pairing $\langle \cdot, \cdot \rangle$ only to keep the presentation compact; all regime/rail statements should be read componentwise in the multi-coupling case.

4 RG instance: scale-dependent readout

Definition 13.2 (Scale (RG) instance $\mathbf{Inst}_{\mathrm{RG}}(\mu)$)**.** Fix a reference scale $\mu > 0$ and a renormalization convention (scheme) as instance data. $\mathbf{Inst}_{\mathrm{RG}}(\mu)$ denotes the operational readout that maps fixed microscopic backbone data to scale-dependent effective couplings and parameters by coarse-graining/decimation above μ. The output is a collection of effective parameters (e.g. $g(\mu)$ or $\alpha(\mu)$) used in rail-level predictions at scale μ.

Remark 13.3 (RG as a representation-level protocol (template view))*.* The RG instance may be read as a representation-protocol instance: a coarse-graining projection Π_μ together with a scheme-dependent base weight μ_X induces the readout law $\mu_R = (\Pi_\mu)_\# \mu_X$. The running coupling ($g(\mu)$ or $\alpha(\mu)$) is an output of this protocol at scale μ, not a change of pre-instance structural invariants.

Remark 13.4 (Energy scale versus coarse-graining length)*.* When the RG instance is implemented by coarse-graining in physical space, the natural identification is $\mu \simeq 1/\ell$, where ℓ is the coarse-graining length. Thus, in the present UV convention, increasing μ corresponds to probing shorter distances. Equivalently, with $t := \ln \mu$, one has $\frac{d}{dt} = \mu \frac{d}{d\mu} = -\ell \frac{d}{d\ell}$. If one instead parametrizes by $s := \ln \ell$ (an IR flow parameter), the beta vector field changes sign.

Remark 13.5 (Scalar versus multi-parameter coupling space)*.* For readability we write a single effective coupling $g(\mu) \in \mathbb{R}$. In general, an RG instance outputs a vector of effective parameters $g(\mu) \in \mathcal{P}$ (a finite-dimensional coupling space), and β is a vector field on $\mathcal{P}$. Any fixed norm on $\mathcal{P}$ may be used in regime indicators/gates; all such norms are equivalent on bounded sets.

Proposition 13.2 (Running coupling as a scale-instance output)**.** *Fix the backbone and* $\mathbf{Inst}_{\mathrm{gauge}}$*. Let* $\mathbf{Inst}_{\mathrm{RG}}(\mu)$ *be a scale instance producing an effective coupling* $g(\mu)$*. Then* $\mu \mapsto g(\mu)$ *is, by construction, a rail/scale-instance output and does not represent a change of pre-instance structural invariants (principal symbol, cone, kernel charges) within the admissible regime.*

Proof. This is immediate from the definition of $\mathbf{Inst}_{\mathrm{RG}}(\mu)$ as a coarse-graining/normalization protocol that fixes a scheme and a reference scale and outputs effective parameters. □

Theorem 13.3 (Asymptotic freedom from a one-loop negative beta function)**.** *Assume* $\mu \mapsto g(\mu)$ *satisfies an RG/Callan–Symanzik flow* (13.1)

$$\mu \frac{dg}{d\mu} = \beta(g). \tag{13.1}$$

Assume the one-loop form

$$\beta(g) = -b_0 g^3 + O(g^5), \quad b_0 > 0,$$

for sufficiently small g *(the sign condition* $b_0 > 0$ *is determined by the chosen gauge/matter content). Then* $g(\mu)$ *decreases along the UV direction and, for sufficiently large* μ*,*

$$0 < g(\mu) \;\le\; \frac{1}{\sqrt{b_0 \log(\mu/\mu_0) + g(\mu_0)^{-2}}}, \tag{13.2}$$

in particular $g(\mu) \to 0$ *as* $\mu \to \infty$ *(asymptotic freedom). See e.g. [2, 3, 4, 5].*

Proof. Let $t = \log \mu$. Then the RG equation becomes $dg/dt = \beta(g)$. Fix $g_\star > 0$ so that the remainder satisfies $|O(g^5)| \le \frac{b_0}{2} g^3$ for all $0 < g \le g_\star$. On any interval where $0 < g(t) \le g_\star$, one has

$$\frac{dg}{dt} \le -\frac{b_0}{2} g^3, \qquad \text{hence} \qquad \frac{d}{dt}\Big(\frac{1}{g^2}\Big) = -2g^{-3}\frac{dg}{dt} \ge b_0.$$

Integrating from $t_0 = \log \mu_0$ to $t = \log \mu$ yields

$$\frac{1}{g(t)^2} \;\ge\; \frac{1}{g(t_0)^2} + b_0(t - t_0) \;=\; \frac{1}{g(\mu_0)^2} + b_0 \log(\mu/\mu_0),$$

which is equivalent to (13.2). In particular $g(\mu)$ is nonincreasing on the UV regime and tends to 0 as $\mu \to \infty$. □

5 Conclusion

Summary (closure statements).

- **Placement.** Gauge forces are encoded as an operator instance $\mathbf{Inst}_{\text{gauge}}$ on the fixed backbone; running couplings are RG-instance outputs.
- **RG is an instance output.** The RG instance is a coarse-graining readout Π_μ with pushforward weight $\mu_R = (\Pi_\mu)_{\#}\mu_X$; running couplings are protocol outputs, not changes of structural invariants.
- **principal operator normal form.** Under the admissibility contract, the gauge kinetic principal operator is canonically represented by the Yang–Mills term $\int \langle F, F\rangle$.
- **Scope boundary.** S0-admissible eliminations preserve the principal operator (cones/c_*); RG-instances are scale/scheme dependent readouts by design.

A Scope boundary: admissible elimination (S0) vs RG scale instance

Admissible eliminations (Part VII, S0) are representative changes within the fixed-density contract that preserve the principal operator. The RG scale instance $\mathbf{Inst}_{\text{RG}}(\mu)$ is a rail-level coarse-graining readout whose outputs are explicitly scale/scheme dependent.

S0 admissible elimination	**RG scale instance $\mathbf{Inst}_{\text{RG}}(\mu)$**
Representative change; preserves principal symbol/cones (c_*).	Rail readout; produces $g(\mu)$, $\alpha(\mu)$ as scale-dependent outputs.
Order-0 modifications only; no principal operator change.	May change effective parameters and composite operators as outputs.
Structural simplification within regime.	Quantitative prediction at a declared scale (scheme-dependent).

Remark 13.6 (S0 vs RG in the series contract). In the series contract, S0 concerns principal operator-preserving representative selection (idempotent readout within a fixed regime), whereas the RG instance concerns scale-dependent coarse-graining (projection with pushforward measure). Their domains are disjoint by design.

B Rail dossier (internal): QCD running as an RG-instance readout

This appendix is an internal rail specification (no new claims). Fix a scheme (recommended: $\overline{\text{MS}}$) and one anchor value $\alpha_s(\mu_0)$.

Normalization note (avoid b_0 ambiguity). In the main text we use the 1-loop normalization $\beta(g) = \dfrac{dg}{d\ln\mu} = -b_0 g^3 + O(g^5)$. With $\alpha_s := g^2/(4\pi)$ this implies $\dfrac{d\alpha_s}{d\ln\mu} = -8\pi b_0\,\alpha_s^2 + O(\alpha_s^3)$. Equivalently, writing the textbook coefficient $\beta_0 := 16\pi^2 b_0$, one has $\dfrac{d\alpha_s}{d\ln\mu} = -(\beta_0/2\pi)\,\alpha_s^2 + \cdots$.

At 1-loop (dropping $O(\alpha_s^3)$) the RG-instance output satisfies

$$\frac{d\,\alpha_s}{d\ln\mu} = -8\pi b_0\,\alpha_s^2, \qquad \alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + 8\pi b_0\,\alpha_s(\mu_0)\ln(\mu/\mu_0)}.$$

Pass/Fail (minimal). AF pass if $\alpha_s(\mu)$ decreases with μ in the declared window when $b_0 > 0$; refine by adding b_1 (2-loop) and threshold matching as gates.

Refinement gates (optional). Upgrade the rail by adding (i) the 2-loop term $\beta(g) = -b_0 g^3 - b_1 g^5$ and (ii) threshold matching when μ crosses quark masses, treating n_f updates and matching conditions as explicit gates.

Level	Checklist (what is checked)
L0 (1-loop shape)	$1/\alpha_s(\mu)$ vs. $\ln\mu$ is approximately linear after fixing one anchor $\alpha_s(\mu_0)$ (slope $\approx 8\pi b_0 = \beta_0/(2\pi)$ in the present normalization).
L1 (2-loop gate)	Add the b_1 term; re-check the curvature of $1/\alpha_s(\mu)$ vs. $\ln\mu$ within the declared window.
L2 (threshold gate)	If μ crosses quark thresholds, update n_f and apply matching conditions; otherwise declare out-of-scope.

Glossary (series meta-language)

- **Structural Layer**: the fixed backbone (primitive E plus admissible equivalences) and the admissible readings (R1/R2/R3).
- **Reading (R1/R2/R3)**: operations applied to the same backbone: R1 (stationarity/EL), R2 (gradient/dissipative), R3 (minimal time-completion/conservative).
- **Elimination**: a representation-level representative selection step (e.g. Schur/dual-Schur reduction) that preserves the declared principal symbol / principal operator lock.
- **Representation instance**: a concrete protocol choice specifying a readout/retention map and base measure.
- **Representation Layer**: protocol data, typically $(\Pi, \mu, \mathcal{G}, \eta)$ where Π is a readout map, μ a base measure, and $(\mathcal{G}, \eta)$ update/event/summary rules.
- **Notation lock ($\mathcal{G}$ vs. G)**: G is reserved for structural variables/operators; $\mathcal{G}$ denotes protocol/update rules in instances.
- **Gate / regime gate**: an explicit validity condition (bounds, coercivity, smallness, slow-variation, etc.) delimiting the regime where a closure statement is licensed.
- **Stopping rule**: an explicit declared exit condition (when a regime/rail no longer applies).
- **Rail**: an optional validation pipeline (pass/fail + diagnostics) attached after fixing an instance; not part of the structural theorem statements.
- **Time instance alias**: a declared synchronization/parameterization protocol used to interpret time variables across regimes.

Scope. All statements are sectorial/regime-level within the declared regime and admissibility gates, and are subject to explicit stopping rules where readout instances are used; no global continuation beyond regime exit is claimed.

References

[1] K. G. Wilson, Renormalization group and critical phenomena, Phys. Rev. B (1971).

[2] C. G. Callan, Jr., Broken scale invariance in scalar field theory, Phys. Rev. D (1970).

[3] K. Symanzik, Small-distance behaviour in field theory, Commun. Math. Phys. (1970).

[4] D. J. Gross and F. Wilczek, Ultraviolet behavior of non-Abelian gauge theories, Phys. Rev. Lett. (1973).

[5] H. D. Politzer, Reliable perturbative results for strong interactions?, Phys. Rev. Lett. (1973).