Variational Backbone and Regime Closures V:

Mass from Variational Spectra and Higgs as an Effective Instance

Schur complement, projector-valued order-0 terms, and regime closure

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January 2026

Abstract

This paper fixes a structural mass mechanism within the series contract "one fixed spatial density $e(\cdot)$ + three operational readings (R1/R2/R3)". Mass is defined as order-0 spectral data of the R1 Hessian at an R2-selected vacuum, equivalently as dispersion data of the R3 time-completed representative. For the canonical first-derivative backbone, eliminating the closed variable $F := D_I I_I$ induces, on the open sector, a Schur complement correction of the form $-(\eta^2/\alpha)P_{\mathrm{Ran}}$: an order-0 projector term that preserves the Laplace-type principal operator and shifts active channels. Elimination and the resulting Schur/projector term are representation-level implementations of a declared readout/retention protocol (Part IV); they do not modify the fixed backbone or its principal operator. After KG normalization, the dimensionless shift is expressed by the series invariant $\eta^2/(\alpha\beta) = \kappa^2$. We state the operator-theoretic gates behind "order-0" and self-adjointness, demonstrate de Rham channel selectivity (exact-only action) via a one-page torus computation, and position Higgs parameters as a representation-level factorization of the extracted mass matrix.

Protocol usage. The representation-protocol vocabulary is formalized in Part IV; here we use only the minimal notion of representative choice (readout/retention and induced weights/updates). Any additional closure/coupling slot is stated explicitly when used. Elimination/Schur/projector procedures are representation-level operations induced by the declared protocol and never modify the Structural Layer.

Self-contained review note. This manuscript is intended to be *self-contained* for peer review. Companion parts of the series are cited only for context and do not supply any result required for the arguments below. Whenever a series-level convention is used (equivalences, instance vocabulary, gates/rails), it is restated locally or declared explicitly.

Meta-language. Nonstandard series terms (reading, instance alias, gate, rail, principal operator lock, etc.) are defined in the *Series meta-language glossary* appended to this manuscript.

Scope. All claims are *sectorial/regime-level* unless explicitly stated: validity and breakdown are controlled by declared regime gates and stopping rules, and no global completeness or continuation is asserted beyond regime exits.

Series pipeline (structure at a glance). Primitive E and constitutional axioms → reading (R1/R2/R3) → elimination (Schur/dual-Schur) → representation instances → (optional) rails for validation. Companion parts of the series develop: (i) metric-channel regime templates, (ii) field-sector regime templates (KG→Sch→HJ), (iii) a time-instance module, and (iv) sectorial closure dossiers (classical/GR/QM/UV), each stated to be review-independent.

Companion parts. Other parts of *Variational Backbone and Regime Closures* may be consulted for background only (see the series map). They do not supply any result required for the arguments in the present manuscript. Dedicated validation rails (cosmology/lensing/data comparison) and optional phenomenology instances (e.g. PPN readouts) are deferred to follow-up dossiers. No step in this manuscript depends on an unpublished argument.

Series scope. All statements are sectorial/regime-level within the declared regime and admissibility gates, and are subject to explicit stopping rules where readout instances are used; no global continuation beyond regime exit is claimed.

Axiomatic provenance (no external postulates added). This paper assumes only the constitutional axioms (A1–A7) and constraints (C1–C8) fixed in Part I (Part I). No foundational postulate from external theories (GR/QM/QFT) is adopted as an additional axiom. Any extra condition introduced below is explicitly declared as a *instance* (convention), a *gate* (regime assumption), or a *rail* (data/units interface), with scope and stage stated where used.

Elimination is a representation-level implementation (not a backbone modification).
In this series, *elimination* means a representation-level representative selection (Schur or dual-Schur complement) performed once a retention channel is declared; it does not commit to any dataset-level rail/readout instance.
It is neither a physical time process nor an observation/measurement.

Remark. Elimination is a representation-level representative selection step (Representation Layer; defined in Part IV): it implements the rule "retain only the observable/retained channel"; the backbone and readings remain structural.

Analytic background (non-normative). For standard background on elliptic operators and Hodge decomposition, see e.g. [1, 6]. For gradient-flow methods and Lojasiewicz–Simon type inequalities used to justify stability/gate statements, see e.g. [3, 4, 5].
Keywords: mass spectrum; second variation; Schur complement; projector; Laplace principal operator; Klein–Gordon regime; Higgs positioning; model instances.
Series context. Parts I–III fix the first-derivative backbone and close the R2/R3 corridors. Part IV defines observational projection and instances. Part V defines mass as vacuum Hessian spectral data and precedes Part VI (dual-Schur closed curvature) and Part VII (elimination duality and structural invariants); validation rails are deferred to a post-Part XII rail dossier.

Mass extraction protocol (registered readout; 7 steps).
(1) Fix the backbone representative and a reading (R1 for Hessian data; R3 for dispersion data).
(2) Declare the open/closed split and the physical open subspace $\mathcal{H}_R^{\mathrm{phys}}$ (mean-zero/gauge slice or stabilized variant).
(3) Fix a vacuum $q^\star$ (R2-selected stationary point) and form the block Hessian at $q^\star$.
(4) Perform representation-level elimination (Schur complement) of the closed block under the admissibility gates (self-adjointness, positivity).
(5) Read off the order-0 spectral shift on the active channels (projector term) and record the invariant coefficient $\kappa^2 = \eta^2/(\alpha\beta)$.
(6) Convert to KG-normalized units when needed and locate the extracted spectrum as the mass readout.
(7) If desired, factorize the extracted matrix into an effective Higgs parameterization (an instance-level re-expression, not a new structural postulate).

Series interface clause (I→XII)

Primitive object (density fixed). We fix one local C^2 scalar functional

$$E[g, I_R, I_I] = \int_M e(x, g, I_R, I_I, \nabla I_R, D_I I_I)\ \mathrm{dvol}_g, \tag{5.1}$$

defined on the configuration variables (g, I_R, I_I). **Density fixed** means: the *functional form* of the local density $e(\cdot)$ is held fixed across readings R1/R2/R3.

Derivation vs. reading. Within each reading we take the ordinary Fréchet derivative of E with respect to (g, I_R, I_I). *Changing the regime* means changing the **operation** applied to the same fixed E: stationarity (R1) vs. gradient descent (R2) vs. conservative time completion (R3).

Operator typing lock (series-wide; fixed for consistency). The interface operator is first-order and typed as

$$D_I : \Gamma(E_I) \to \Gamma(E_R), \qquad F := D_I I_I \in \Gamma(E_R),$$

so the minimal bilinear interface term $\langle I_R,\, F\rangle$ is well-typed. (Formerly this structural variable was denoted G in Parts I–III; we write F here to avoid collision with G_N.) We assume D_I admits a formal adjoint D_I^* with respect to g and dvol_g. **Operator instance requirement:** each paper must specify $(E_I, E_R, D_I, \ker(D_I)$ treatment).

Kernel / gauge convention (closed sector). If the density depends on I_I only through $F = D_I I_I$, then I_I is understood modulo $\mathrm{Ker}(D_I)$. Equivalently, fix a representative by a gauge/mean-zero slice so that F is the effective closed primitive.

Zero-mode convention (open sector). The open sector uses a default zero-mode removal (mean-zero / gauge fix) so that a Poincaré-type control holds, or equivalently one may add a small mass stabilization as a lower-order modification. Whenever estimates rely on a Poincaré constant $C_P(g)$, one must declare either: (i) an admissible metric class with uniform C_P, or (ii) the mass-stabilized convention.

Controlled equivalences ("up to"). Representative statements are only asserted up to the following moves: (i) boundary/divergence terms (on closed M they do not affect interior EL), (ii) constant rescalings / time reparametrizations, (iii) lower-order stabilizations preserving coercivity and principal symbol, (iv) invertible linear fibre reparametrizations preserving the quadratic backbone class.

Three readings (one E, three operations).

- **(R1) Stationary:** impose $\delta E = 0$ (Euler–Lagrange stationarity/constraints).
- **(R2) Dissipative:** choose an L^2-type pairing (Hilbert completion) in the chosen direction and define the gradient by Riesz representation; evolve by $\partial_\tau u = -\nabla E(u)$.
- **(R3) Conservative:** perform a minimal time-completion *template*: add a positive quadratic time-kinetic term, keep the spatial potential exactly $-E$ (same fixed density $e(\cdot)$), and introduce no new spatial derivative orders.

Times. τ denotes dissipative (R2) time; t denotes conservative (R3) time.

Metric direction conventions (when used). In the metric direction we use the standard $L^2(g)$ pairing on $\Gamma(\mathrm{Sym}^2 T^*M)$. Metric evolutions are interpreted modulo diffeomorphisms; DeTurck gauge fixing and normalizations are part of the R2 closure.

Regime and contract (Parts I–III). Unless stated otherwise, results are established on an admissible regime $\mathcal{U}$ where: g is smooth and nondegenerate; D_I^* is well-defined; coercivity assumptions hold. Degenerate metrics and non-smooth projections are deferred to Part IV.

Contents

1 Notation and conventions

We use the PDE sign convention for the Laplacian, $\Delta = -\nabla^*\nabla$, so that $-\Delta \geq 0$ on the declared mean-zero/gauge slice. An "instance" refers to a representation/protocol choice as formalized in Part IV. The closed sector is represented by $F := D_I I_I$ and treated modulo $\mathrm{Ker}(D_I)$ (Series interface clause). P_{Ran} denotes the $L^2(g)$-orthogonal projector onto $\mathrm{Ran}(D_I)$ (or its canonical formula (5.21) when available). Although P_{Ran} is generally nonlocal on the chosen completion (it involves a pseudoinverse), it is an order-0 operator and therefore does not alter the locked Laplace-type principal operator; nonlocality enters only through bounded spectral shifts (mass-shell readouts). The dimensionless coupling

$$\kappa := \frac{\eta}{\sqrt{\alpha\beta}} \tag{5.2}$$

is invariant under the declared backbone equivalences (Part I; series contract). In the KG-normalized dispersion form (Section 10), the Schur coefficient appears as $\eta^2/(\alpha\beta) = \kappa^2$. Spectral data are evaluated on the physical open subspace $\mathcal{H}_R^{\rm phys}$ (Definition 5.3). The vacuum mass gap is the bottom of the *positive* spectrum $\lambda^+_{\min}(q^\star)$ and $\omega_{\min}(q^\star) = \sqrt{\lambda^+_{\min}(q^\star)}$ (Definition 5.4).

Remark 5.1 (Why κ is invariant under rescalings). Under a general constant rescaling of the open and closed variables, we may pass to new representatives $\widetilde{I}_R := s\, I_R$ and $\widetilde{F} := t\, F$ $(s,t \neq 0)$, equivalently $I_R = \widetilde{I}_R/s$ and $F = \widetilde{F}/t$. Then the canonical representative (5.3) retains its form with transformed coefficients $\widetilde{\beta} = \beta/s^2$, $\widetilde{\alpha} = \alpha/t^2$, and $\widetilde{\eta} = \eta/(st)$. Hence

$$\widetilde{\kappa} = \frac{\widetilde{\eta}}{\sqrt{\widetilde{\alpha}\,\widetilde{\beta}}} = \frac{\eta/(st)}{\sqrt{(\alpha/t^2)(\beta/s^2)}} = \frac{\eta}{\sqrt{\alpha\beta}} = \kappa.$$

The open-only rescaling used elsewhere is the special case $t = 1$.

2 Scope and statement

Principal operator lock $\Rightarrow$ order-0. The open sector has a Laplace-type principal part locked by the series backbone; therefore mass can only enter as an order-0 (i.e. bounded on L^2 under the declared instances) term in the linearized operator at a stationary background.

Forced Schur term. We emphasize that the closed sector is treated in the curvature variable $F := D_I I_I$ (rather than in I_I itself); accordingly the closed quadratic block in (5.3) is $(\alpha/2)\|F\|^2$ and the corresponding Hessian block is $\alpha\,\mathrm{Id}$ on $\mathrm{Ran}(D_I)$ on the declared completion. If the density contains a coercive closed quadratic term $\frac{\alpha}{2}\|F\|^2$ $(\alpha > 0)$ and a bilinear interface $\eta\langle I_R, F\rangle$ $(\eta \neq 0)$, then eliminating F forces the Schur complement $-(\eta^2/\alpha)P_{\rm Ran}$ in the open operator. Higgs-language parameters are treated as a representation-level factorization of this order-0 spectral data.

Contributions (novelty within the series).

1. **Definition lock:** mass is fixed as order-0 Hessian spectrum at an R2-selected vacuum, and as the same spectrum read by R3 dispersion.
2. **Invariant coefficient:** in KG-normalized form the dimensionless Schur shift is $\eta^2/(\alpha\beta) = \kappa^2$, compatible with the Part I series contract (declared equivalences).
3. **Channel selectivity + example:** for the de Rham instance, the Schur shift acts only on exact modes; a flat-torus Fourier computation makes the shifted/unchanged channels explicit (see Section B).
4. **Higgs positioning:** Higgs-sector parameters are treated as a representation-level reparametrization/factorization of the extracted mass matrix (compatibility dictionary only).

3 Canonical representative and linearization

3.1 Field-sector representative

We restrict to a minimal representative of the canonical first-derivative class:

$$E_{\rm fld}[g, I_R, I_I] = \int_M \Big(\frac{\beta}{2}|\nabla I_R|_g^2 + \frac{\alpha}{2}|F|_g^2 + \eta\,\langle I_R, F\rangle + V(I_R)\Big)\,\mathrm{dvol}_g, \qquad F := D_I I_I, \tag{5.3}$$

with $\alpha, \beta > 0$ and $\eta \in \mathbb{R}$.

Remark 5.2 (Optional EH sector). One may adjoin an EH-leading metric sector as in Part II. Here g is fixed unless stated otherwise.

3.2 Stationary background and perturbations

Fix a stationary background $(g, I_R^\star, I_I^\star)$ in the R1 reading (after gauge conventions):

$$\delta E_{\text{fld}}[g, I_R^\star, I_I^\star] = 0.$$

Write perturbations

$$I_R = I_R^\star + u, \qquad I_I = I_I^\star + w, \qquad F = F^\star + v, \quad v := D_I w.$$

We treat the closed sector in the gauge-invariant variable F and remove $\mathrm{Ker}(D_I)$ modes.

4 Vacuum selection (R2)

4.1 R2 gradient flow and dissipation

Let q denote the field configuration (with kernel/zero-mode conventions enforced): $q := (I_R, I_I)$, equivalently (I_R, F) with $F := D_I I_I$. Fix the L^2 pairing used in Parts I–II. The R2 reading defines the gradient flow

$$\partial_\tau q(\tau) = -\nabla E_{\text{fld}}(q(\tau)). \tag{5.4}$$

Along smooth solutions,

$$\frac{d}{d\tau} E_{\text{fld}}(q(\tau)) = -\|\nabla E_{\text{fld}}(q(\tau))\|_{L^2}^2 \leq 0. \tag{5.5}$$

Assume E_{fld} is bounded below on the admissible class. Then $E_{\text{fld}}(q(\tau))$ converges as $\tau \to \infty$ and

$$\int_0^\infty \|\nabla E_{\text{fld}}(q(\tau))\|_{L^2}^2 \, d\tau < \infty. \tag{5.6}$$

4.2 ω-limit criticality

Proposition 5.1 (ω-limit points are R1 stationary)**.** *Let $q(\tau)$ solve* (5.4)*. Suppose there exists a sequence $\tau_n \to \infty$ such that $q(\tau_n) \to q^\star$ in a topology for which ∇E_{fld} is continuous. Then $\nabla E_{\text{fld}}(q^\star) = 0$.*

Proof. By (5.6) there exists $\tau_n \to \infty$ with $\|\nabla E_{\text{fld}}(q(\tau_n))\|_{L^2} \to 0$. Passing to the limit using continuity yields $\nabla E_{\text{fld}}(q^\star) = 0$. □

Remark 5.3 (Hodge channel split: massless propagation vs mass-generating channel). Under the de Rham operator instance (so that an L^2 Hodge decomposition applies), the open sector admits the canonical splitting

$$I_R = I_R^{\text{ex}} \oplus I_R^{\text{coex}} \oplus I_R^{\text{harm}}.$$

The Schur/projector correction produced by the closed sector elimination is order-0 and acts only on the *active* (exact) channel selected by the declared projector $\mathcal{P}$; this is the mass-generating channel. By contrast, the coexact/harmonic channels may remain purely propagating (massless at the principal operator level) or topological (no dissipative loss under R2), depending on the admissible gates and instance choice.

4.3 Convergence to a single limit (Lojasiewicz–Simon regime)

To make vacuum selection reproducible, we adopt the standard analytic-gradient regime.

Assumption 5.1 (Lojasiewicz–Simon inequality)**.** Near any critical point $q^\star$ of E_{fld}, there exist constants $C > 0$ and $\theta \in (0, 1/2]$ such that

$$\left|E_{\text{fld}}(q) - E_{\text{fld}}(q^\star)\right|^{1-\theta} \leq C \left\|\nabla E_{\text{fld}}(q)\right\|_{L^2} \tag{5.7}$$

holds for all q in a neighborhood of $q^\star$.

Theorem 5.2 (R2 selects a single stationary limit)**.** *Assume the R2 trajectory $q(\tau)$ is precompact in the topology of Proposition 5.1 and that Assumption 5.1 holds. If $q(\tau_n) \to q^\star$ for some $\tau_n \to \infty$, then $q(\tau) \to q^\star$ as $\tau \to \infty$.*

Proof. This is the standard Lojasiewicz–Simon argument for analytic gradient flows. Energy dissipation (5.5) implies $E_{\text{fld}}(q(\tau))$ converges and $\nabla E_{\text{fld}}(q(\tau)) \in L^2([0, \infty))$. Near a limit critical point $q^\star$, the Lojasiewicz–Simon inequality (5.7) converts small energy gap into control of the gradient, yielding integrability of the speed and hence convergence of $q(\tau)$ to $q^\star$. See [4, 5] for details. □

Remark 5.4. Theorem 5.2 is a standard consequence of (5.5) and (5.7); see [4, 5]. In this series it is used as a regime closure for vacuum selection.

Remark 5.5 (Role of the R2 gate). Assumption 5.1 and precompactness are used only to make the R2 limit unique within a basin, i.e. to select a reproducible vacuum. All R1 statements below (Hessian block form, Schur complement, order-0 mass data) apply to any stationary point satisfying the declared stability/admissibility gate; the R2 reading is a selection protocol, not an additional mass source.

4.4 Vacuum definition

Definition 5.1 (Vacuum (R2-selected stable stationary point))**.** A *vacuum* is a stationary point $q^\star$ (R1: $\nabla E_{\text{fld}}(q^\star) = 0$) such that:

1. (**R2 selection**) there exists a nontrivial basin $\mathcal{B}(q^\star)$ of initial data q_0 for which the R2 flow (5.4) converges to $q^\star$,
2. (**Stability**) the Hessian $H(q^\star) = D^2 E_{\text{fld}}(q^\star)$ is nonnegative on the declared constraint subspace (after kernel/zero-mode conventions).

Remark 5.6 (Phase structure). Distinct vacua correspond to distinct basins $\mathcal{B}(q^\star)$ and may yield distinct Hessian spectra. This is not parameter insertion; it is basin-dependent phase structure of one fixed E.

4.5 Convention: all "$\star$" data are evaluated at a vacuum

Henceforth we fix $q^\star$ to be an R2-selected vacuum in the sense of Definition 5.1. All quantities with a "$\star$" subscript in Sections 5–15 are evaluated at $q^\star$.

5 Hessian and Schur complement (R1)

5.1 Quadratic model at the background

Let $V''_\star := D^2 V(I_R^\star)$. The quadratic approximation of (5.3) in (u, v) is

$$Q(u, v) = \frac{1}{2} \int_M \left(\beta |\nabla u|_g^2 + \langle u, V''_\star u\rangle + \alpha |v|_g^2 + 2\eta \langle u, v\rangle\right) \mathrm{dvol}_g, \tag{5.8}$$

Schur elimination in one line (expository). Writing the quadratic form (5.8) in block form, in the present setting, A is the Hessian block on the retained (open) variables, C is the Hessian block on the eliminated variables, and B is the mixed block (the bilinear coupling) between them (all taken in the declared L^2 pairing). $\frac{1}{2}\langle (u,v), \left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right)(u,v)\rangle$, the elimination of v means: *for each u, choose v (in the admissible constraint set) that minimizes $Q(u,v)$*. Formally, on the unconstrained block this gives $v_{\mathrm{opt}}(u) = -C^{-1}B^*u$ and the effective open quadratic form $Q_{\mathrm{eff}}(u) = \frac{1}{2}\langle u, (A - BC^{-1}B^*)u\rangle$ (the Schur complement).

Interpretation (why $-BC^{-1}B^*$ is not an added term).
The Schur correction is the *unique* order-0 response induced by eliminating the complementary block under the declared pairing and admissibility gates. It preserves the Laplace-type principal operator (the $|\xi|^2$ part) and only shifts the active retained channels by a bounded projector-valued term.

When v is constrained (here $v \in \mathrm{Ran}(D_I)$), the same formula holds after replacing C^{-1} by the inverse on the admissible subspace, equivalently inserting the projector onto $\mathrm{Ran}(D_I)$; this is the origin of projector-valued order-0 corrections under the constraint $v \in \mathrm{Ran}(D_I) \subset \Gamma(E_R)$.

Definition 5.2 (Mass operator (R1))**.** Let H denote the Hessian operator associated with Q on a chosen Hilbert completion. The *effective open Hessian* is the Schur complement obtained by eliminating v over its admissible constraint set. The *mass operator* is the order-0 part of this effective open Hessian (projector-valued in general).

In the Laplace-type class, we call "mass" the bounded/order-0 contribution to the effective operator relative to the fixed principal operator $-\beta\Delta$.

5.2 General Schur formula (Hilbert-space statement)

Let $\mathcal{H}_R$ and $\mathcal{H}_I$ be Hilbert spaces for the open and closed variations, and let

$$Q(u,v) = \frac{1}{2}\langle u, Au\rangle_{\mathcal{H}_R} + \langle u, Bv\rangle_{\mathcal{H}_R} + \frac{1}{2}\langle v, Cv\rangle_{\mathcal{H}_I},$$

where A is self-adjoint on $\mathcal{H}_R$, C is self-adjoint and strictly positive on the admissible subspace of $\mathcal{H}_I$, and B is a bounded operator.

Theorem 5.3 (Schur complement as elimination)**.** *Assume $C \geq c\,I$ on the admissible closed subspace for some $c > 0$. Then for each u, the minimizer of $v \mapsto Q(u,v)$ is $v_{\min}(u) = -C^{-1}B^*u$, and the reduced quadratic form is*

$$Q_{\mathrm{eff}}(u) = \frac{1}{2}\langle u, (A - BC^{-1}B^*)u\rangle_{\mathcal{H}_R}.$$

Consequently, the effective open Hessian is

$$H_{\mathrm{eff}} = A - BC^{-1}B^*. \tag{5.9}$$

Proof. Fix u and view $Q(u,\cdot)$ as a strictly convex quadratic functional in v on the admissible closed subspace. Setting its first variation to zero gives $B^*u + Cv = 0$, hence $v_{\min}(u) = -C^{-1}B^*u$. Substituting this into Q and using self-adjointness of C yields $Q_{\mathrm{eff}}(u) = \frac{1}{2}\langle u, (A - BC^{-1}B^*)u\rangle$, i.e. (5.9). □

Remark 5.7 (Why this forces mass). The series lock fixes the principal part of A (Laplace type). Therefore $BC^{-1}B^*$ can only modify lower order. If $B \neq 0$, then $BC^{-1}B^* \neq 0$ and an order-0 spectral shift is unavoidable on the active channel.

5.3 Dual Schur complement and critical equivalence (preview of Part VI)

The reduction in Theorem 5.3 eliminates the closed variable v and produces an effective *open* Hessian $H_{\mathrm{eff}} = A - BC^{-1}B^*$. In the complementary admissible regime where the open block is invertible (so A^{-1} exists as a bounded operator), one may also eliminate the open variable and obtain the *dual* (closed) Schur complement

$$H_{\mathrm{cl,eff}} \;:=\; C - B^*A^{-1}B. \tag{5.10}$$

This dual reduction is the starting point of Part VI (dual-Schur closed-curvature structural paper) and is responsible for the forced k^{-2}-type scale-response shape discussed there.

Theorem 5.4 (Critical equivalence of Schur and dual-Schur reductions)**.** *Let $\mathcal{H}_R, \mathcal{H}_I$ be Hilbert spaces and consider the block Hessian*

$$H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

associated with Q. Assume $C \geq c\,I$ on the admissible closed subspace for some $c > 0$ and $A \geq a\,I$ on the admissible open subspace for some $a > 0$, so that both C^{-1} and A^{-1} exist as bounded operators on the declared slices. Define the Schur complements

$$H_{\mathrm{eff}} := A - BC^{-1}B^*, \qquad H_{\mathrm{cl,eff}} := C - B^*A^{-1}B.$$

Then the following are equivalent:

1. *H has a nontrivial null vector on the admissible product subspace (loss of coercivity of the full quadratic form);*
2. *H_{eff} has a nontrivial kernel (open gap closing / onset of instability in the R3 reading);*
3. *$H_{\mathrm{cl,eff}}$ has a nontrivial kernel (closed criticality / dual-Schur threshold).*

Moreover,

$$u \in \mathrm{Ker}(H_{\mathrm{eff}}) \iff \big(u,\, -C^{-1}B^*u\big) \in \mathrm{Ker}(H),$$
$$v \in \mathrm{Ker}(H_{\mathrm{cl,eff}}) \iff \big(-A^{-1}Bv,\, v\big) \in \mathrm{Ker}(H).$$

Proof. If C is invertible on the admissible closed slice, then the Schur identity gives

$$H\begin{pmatrix} u \\ v \end{pmatrix} = 0 \iff \begin{cases} Au + Bv = 0, \\ B^*u + Cv = 0, \end{cases} \iff \begin{cases} (A - BC^{-1}B^*)u = 0, \\ v = -C^{-1}B^*u, \end{cases}$$

which yields the equivalence between (1) and (2) and the first kernel correspondence. Similarly, if A is invertible on the admissible open slice, then

$$H\begin{pmatrix} u \\ v \end{pmatrix} = 0 \iff \begin{cases} u = -A^{-1}Bv, \\ (C - B^*A^{-1}B)v = 0, \end{cases}$$

giving the equivalence between (1) and (3) and the second correspondence. □

Remark 5.8 (Canonical specialization and the open–closed critical threshold). In the canonical representative $C = \alpha I$ on $\mathrm{Ran}(D_I)$ and $B = \eta P_{\mathrm{Ran}}$, so

$$H_{\mathrm{cl,eff}} = \alpha I - \eta^2\, P_{\mathrm{Ran}} A^{-1} P_{\mathrm{Ran}}.$$

Part VI analyzes the induced scale-response and records the closed critical threshold $\eta^2 \lambda_{\max}(P_{\mathrm{Ran}} A^{-1} P_{\mathrm{Ran}}) = \alpha$. By Theorem 5.4, this is *equivalent* (on the same declared instance and vacuum) to open gap closing for $H_{\mathrm{eff}} = A - (\eta^2/\alpha) P_{\mathrm{Ran}}$, i.e. to $\omega_{\min} \to 0$ in Section 12.4.

5.4 Specialization to the canonical backbone

Let P_{Ran} be the $L^2(g)$-orthogonal projector $\Gamma(E_R) \to \mathrm{Ran}(D_I)$. Then (5.8) is of the form in Theorem 5.3 with

$$A = -\beta\Delta + V''_\star, \qquad C = \alpha I \text{ on } \mathrm{Ran}(D_I), \qquad B = \eta P_{\mathrm{Ran}}.$$

Lemma 5.5 (Closed minimizer and reduced form). *For fixed u, the minimizer over $v \in \mathrm{Ran}(D_I)$ is*

$$v_{\min}(u) = -\frac{\eta}{\alpha}\, P_{\mathrm{Ran}} u,$$

and the reduced quadratic form is

$$Q_{\mathrm{eff}}(u) = \frac{1}{2}\int_M \Big(\beta|\nabla u|_g^2 + \langle u, V''_\star u\rangle - \frac{\eta^2}{\alpha}\,|P_{\mathrm{Ran}} u|_g^2\Big)\,\mathrm{dvol}_g. \tag{5.11}$$

Proof. Apply Theorem 5.3 with $C = \alpha I$ on $\mathrm{Ran}(D_I)$ and $B = \eta P_{\mathrm{Ran}} = B^*$. Then $v_{\min}(u) = -C^{-1}B^*u = -(\eta/\alpha)P_{\mathrm{Ran}}u$. Substituting into Q and using $P_{\mathrm{Ran}}^2 = P_{\mathrm{Ran}}$ gives (5.11). □

Proposition 5.6 (Effective open operator). *The effective open Hessian is the operator*

$$L_{\mathrm{eff}} = -\beta\Delta + V''_\star - \frac{\eta^2}{\alpha}\,P_{\mathrm{Ran}}. \tag{5.12}$$

In particular, the Schur term is order-0 (projector-valued, possibly nonlocal) and preserves the Laplace principal operator.

Proof. Differentiate the reduced quadratic form (5.11) with respect to u. The $\beta|\nabla u|^2$ term yields $-\beta\Delta u$ after integration by parts, $\langle u, V''_\star u\rangle$ yields $V''_\star u$, and the projector term yields $-(\eta^2/\alpha)P_{\mathrm{Ran}}u$. Thus the associated self-adjoint operator is (5.12). □

Corollary 5.7 (Inevitability on active channels). *If $\eta \neq 0$ and $\mathrm{Ran}(D_I) \neq \{0\}$, then the Schur term in (5.12) is nonzero. Moreover, on any mode u with $P_{\mathrm{Ran}}u \neq 0$, the order-0 spectral data of L_{eff} differ from those of $-\beta\Delta + V''_\star$. Hence a mass shift is forced on the active channel unless the coupling is turned off ($\eta = 0$) or the mode lies in $\mathrm{Ker}(P_{\mathrm{Ran}})$.*

Proof. If $\mathrm{Ran}(D_I) \neq \{0\}$ then the orthogonal projector $P_{\mathrm{Ran}} \neq 0$. If additionally $\eta \neq 0$, then $(\eta^2/\alpha)P_{\mathrm{Ran}} \neq 0$, so the Schur term is nontrivial. For any u with $P_{\mathrm{Ran}}u \neq 0$, the quadratic form (or Rayleigh quotient) of L_{eff} differs from that of the bare operator $-\beta\Delta + V''_\star$, hence the order-0 spectral data on that active channel are shifted. □

Theorem 5.8 (T3: Closed-sector elimination forces an order-0 mass term). *Assume the canonical first-derivative backbone (Part I, A4/A4') and let $q^\star$ be an R2-selected vacuum. Then eliminating the closed sector variable $v \in \mathrm{Ran}(D_I)$ in the quadratic model (5.8) forces a projector-valued order-0 correction on the open sector:*

$$L_{\mathrm{eff}} = -\beta\Delta + V''_\star - \frac{\eta^2}{\alpha}\,P_{\mathrm{Ran}}, \tag{5.13}$$

as in Proposition 5.6. In particular, if $\eta \neq 0$ and $\mathrm{Ran}(D_I) \neq \{0\}$, the effective operator differs from the bare Laplace-type principal operator on the active channel and therefore carries nontrivial spectral data at order 0 ("variational mass").

Proof. The Schur complement computation gives (5.13) (Proposition 5.6), while Corollary 5.7 yields nontriviality on active modes. □

Remark 5.9 (KG normalization and the dimensionless correction). The invariant (5.2) is the KG-normalized dimensionless strength of the order-0 correction. Equivalently,

$$\kappa^2 = \frac{\eta^2}{\alpha\beta}. \tag{5.14}$$

6 General closed quadratic forms and multicomponent couplings

This section records a minimal extension in which the closed quadratic term is anisotropic (or multicomponent) and the open–closed coupling is an arbitrary order-0 bundle map. The point is structural: *as long as the closed block is coercive and the coupling is nonzero, the Schur complement produces an order-0 correction on the open operator*, hence a forced mass shift on active channels.

6.1 Anisotropic closed metric (bundle endomorphism)

Consider a quadratic representative at a stationary background of the form

$$Q(u,v) = \frac{1}{2}\int_M \Big(\beta|\nabla u|_g^2 + \langle u, V_\star'' u\rangle + \langle v, \mathsf{A}v\rangle + 2\,\langle \mathsf{K}u, v\rangle\Big)\,\mathrm{dvol}_g, \qquad v \in \mathrm{Ran}(D_I), \tag{5.15}$$

where A is a pointwise self-adjoint positive-definite bundle endomorphism on E_R (restricted to the admissible closed subspace) and K is an order-0 bundle map on the open variable. Let P_{Ran} be the $L^2(g)$-orthogonal projector onto $\mathrm{Ran}(D_I)$.

Proposition 5.9 (Anisotropic Schur mass operator). *Assume $\mathsf{A} \ge a_0 I$ pointwise for some $a_0 > 0$. Then the minimizer of $v \mapsto Q(u,v)$ over $v \in \mathrm{Ran}(D_I)$ is*

$$v_{\min}(u) = -\mathsf{A}^{-1} P_{\mathrm{Ran}} \mathsf{K} u,$$

and the reduced quadratic form is

$$Q_{\mathrm{eff}}(u) = \frac{1}{2}\int_M \Big(\beta|\nabla u|_g^2 + \langle u, V_\star'' u\rangle - \langle P_{\mathrm{Ran}}\mathsf{K}u, \mathsf{A}^{-1} P_{\mathrm{Ran}}\mathsf{K}u\rangle\Big)\,\mathrm{dvol}_g.$$

Equivalently, the effective open operator is

$$L_{\mathrm{eff}} = -\beta\Delta + V_\star'' - \mathsf{K}^* \, P_{\mathrm{Ran}}\, \mathsf{A}^{-1}\, P_{\mathrm{Ran}}\, \mathsf{K}. \tag{5.16}$$

In particular, the Schur term is order-0 and preserves the Laplace principal operator.

Proof. For fixed u, $Q(u,\cdot)$ is strictly convex in v on $\mathrm{Ran}(D_I)$ because $\mathsf{A} \ge a_0 I$. Taking the first variation in v gives $\mathsf{A}v + P_{\mathrm{Ran}}\mathsf{K}u = 0$, hence $v_{\min}(u) = -\mathsf{A}^{-1}P_{\mathrm{Ran}}\mathsf{K}u$. Substituting into (5.15) yields the reduced form and identifies the Schur term as $\mathsf{K}^* P_{\mathrm{Ran}} \mathsf{A}^{-1} P_{\mathrm{Ran}} \mathsf{K}$, which is order-0. □

Remark 5.10 (Forced shift on active channels). If $P_{\mathrm{Ran}}\mathsf{K}u \neq 0$ for some admissible u, then the Schur term in (5.16) is nonzero on that channel. Thus, unless the coupling is turned off (effective $\mathsf{K} = 0$ on $\mathrm{Ran}(D_I)$), an order-0 spectral shift—hence a mass shift in the R3 dispersion reading—is forced.

7 Coercivity and spectral lower bounds

The previous section identifies the effective open operator $L_{\mathrm{eff}} = -\beta\Delta + V_\star'' - (\eta^2/\alpha)P_{\mathrm{Ran}}$. Here we record a minimal coercivity statement ensuring that L_{eff} is bounded below on the declared open subspace.

Assumption 5.2 (Poincaré control on the open subspace)**.** On the declared open subspace (mean-zero / gauge-fixed), there exists $\lambda_1 = \lambda_1(g) > 0$ such that

$$\|u\|_{L^2}^2 \le \frac{1}{\lambda_1} \|\nabla u\|_{L^2}^2 \qquad \text{for all admissible } u. \tag{5.17}$$

Assumption 5.3 (Lower bound for the potential curvature)**.** There exists $\mu_0 \in \mathbb{R}$ such that

$$\langle u, V_\star'' u\rangle_{L^2} \ge \mu_0 \|u\|_{L^2}^2 \qquad \text{for all admissible } u. \tag{5.18}$$

Lemma 5.10 (Energy identity)**.** *For any admissible u,*

$$\langle u, L_{\text{eff}} u\rangle_{L^2} = \beta \|\nabla u\|_{L^2}^2 + \langle u, V_\star'' u\rangle_{L^2} - \frac{\eta^2}{\alpha} \|P_{\text{Ran}} u\|_{L^2}^2. \tag{5.19}$$

Moreover, since P_{Ran} is an L^2-orthogonal projection, $\|P_{\text{Ran}} u\|_{L^2} \le \|u\|_{L^2}$.

Proof. Using self-adjointness and integration by parts, $\langle u, -\beta\Delta u\rangle_{L^2} = \beta\|\nabla u\|_{L^2}^2$. The remaining terms are algebraic: $\langle u, V_\star'' u\rangle_{L^2}$ and $-(\eta^2/\alpha)\langle u, P_{\text{Ran}} u\rangle_{L^2} = -(\eta^2/\alpha)\|P_{\text{Ran}} u\|_{L^2}^2$ because P_{Ran} is an orthogonal projector. □

Proposition 5.11 (A sufficient coercivity criterion)**.** *Assume 5.2 and 5.3. Then*

$$\langle u, L_{\text{eff}} u\rangle_{L^2} \ge (\beta\lambda_1 + \mu_0 - \eta^2/\alpha) \|u\|_{L^2}^2 \qquad \textit{for all admissible } u. \tag{5.20}$$

In particular, if $\beta\lambda_1 + \mu_0 > \eta^2/\alpha$, then L_{eff} is bounded below by a positive constant on the declared open subspace.

Proof. Combine (5.19), the contraction $\|P_{\text{Ran}} u\| \le \|u\|$, and (5.18) to obtain

$$\langle u, L_{\text{eff}} u\rangle \ge \beta\|\nabla u\|^2 + (\mu_0 - \eta^2/\alpha)\|u\|^2.$$

Apply (5.17), equivalently $\|\nabla u\|^2 \ge \lambda_1 \|u\|^2$. □

Stability gate G_{vac} (explicit). The Schur shift $-(\eta^2/\alpha)P_{\text{Ran}}$ is a spectral lowering on active channels and can destabilize a stationary point unless the Laplace backbone and/or $V_\star''$ supplies sufficient rigidity. A sufficient regime-level gate is

$$\beta\lambda_1 + \mu_0 > \eta^2/\alpha,$$

which ensures coercivity of L_{eff} on the declared open subspace.

Remark 5.11 (Interpretation). The criterion (5.20) is only sufficient. On modes in $\text{Ker}(P_{\text{Ran}})$ the Schur term vanishes and coercivity can hold under weaker conditions. The point of (5.20) is to provide a regime-level inequality that closes well-posedness for the R3 dispersion and the R2 gradient reading without changing the locked principal symbol.

8 Projector from the operator instance

8.1 Canonical projector formula

On a Hilbert completion where D_I is closed and admits a formal adjoint, a canonical expression is

$$P_{\text{Ran}} = D_I \,(D_I^* D_I)^+\, D_I^*, \tag{5.21}$$

where $(\cdot)^+$ denotes the Moore–Penrose pseudoinverse (inverse on $\text{Ker}(D_I)^\perp$).

Remark 5.12 (Order and nonlocality). The inverse $(D_I^* D_I)^+$ typically yields nonlocality. This does not violate the series lock: it does not change the principal symbol class of the open Laplace principal operator.

8.2 de Rham instance (exact channel)

Instance 5.1 (de Rham instance)**.** Take $E_I = \Lambda^k T^*M$, $E_R = \Lambda^{k+1} T^*M$, $D_I = d$, $D_I^* = d^*$, so $F = dI_I$ is exact.

Then P_{Ran} is the exact Hodge projector

$$P_{\mathrm{Ran}} = P_{\mathrm{ex}} := d\,\Delta_0^{-1}\,d^*, \tag{5.22}$$

with Δ_0^{-1} defined on an admissible slice (kernel/mean-zero convention).

Theorem 5.12 (Exact-channel mass theorem (de Rham instance))**.** *Under Instance 5.1, let $u = u_{\mathrm{ex}} + u_{\mathrm{coex}} + u_{\mathrm{harm}}$ be the Hodge decomposition (orthogonal in $L^2(g)$). Then $P_{\mathrm{ex}} u = u_{\mathrm{ex}}$ and the effective operator splits as*

$$L_{\mathrm{eff}} u = (-\beta\Delta + V_\star'')\,(u_{\mathrm{coex}} + u_{\mathrm{harm}}) + (-\beta\Delta + V_\star'' - \eta^2/\alpha)\,u_{\mathrm{ex}}. \tag{5.23}$$

Equivalently,

$$L_{\mathrm{eff}} \cong (-\beta\Delta + V_\star'' - \eta^2/\alpha)\big|_{\mathrm{ex}} \;\oplus\; (-\beta\Delta + V_\star'')\big|_{\mathrm{coex}\oplus\mathrm{harm}}.$$

Proof. By Hodge theory on a closed manifold, $\Gamma(\Lambda^{k+1} T^*M) = \mathrm{Ran}(d) \oplus \mathrm{Ran}(d^*) \oplus \mathcal{H}^{k+1}$ orthogonally. Since $P_{\mathrm{ex}} = d\Delta_0^{-1} d^*$ is the $L^2(g)$-orthogonal projector onto $\mathrm{Ran}(d)$, it acts as the identity on exact forms and vanishes on coexact and harmonic forms. Substitute this into $L_{\mathrm{eff}} = -\beta\Delta + V_\star'' - (\eta^2/\alpha) P_{\mathrm{ex}}$ to obtain (5.23). □

Remark 5.13 (Topological zero modes). Harmonic modes are controlled by topology: $\dim \mathcal{H}^{k+1} = b_{k+1}(M)$. They are unaffected by the Schur-induced order-0 shift in the de Rham instance.

9 Spectral conventions (gauge and zero-modes)

9.1 Gate: order-0 terms and self-adjointness

In this series, "order-0" means: an operator that is bounded on L^2 (equivalently, a pseudodifferential operator of order 0 on a smooth compact manifold). Orthogonal projectors and zeroth-order bundle endomorphisms fall into this class.

Assumption 5.4 (Functional-analytic gate)**.** Work on the physical subspace $\mathcal{H}_R^{\mathrm{phys}}$ (Definition 5.3) where: (i) $-\Delta$ is self-adjoint with domain $\mathcal{D}(-\Delta)$, (ii) $\mathrm{Ran}(D_I) \subset L^2$ is closed (e.g. for elliptic/Fredholm operator instances on compact M), and P_{Ran} is realized as the L^2-orthogonal projector onto $\mathrm{Ran}(D_I)$ (hence bounded and self-adjoint; order-0 on the declared instances), and (iii) $V_{q^\star}''$ is self-adjoint and bounded on L^2 (or relatively bounded with respect to $-\Delta$ with Kato bound < 1).

Proposition 5.13 (Self-adjointness of L_{eff})**.** *Under Assumption 5.4, the effective operator*

$$L_{\mathrm{eff}}(q^\star) = -\beta\Delta + V_{q^\star}'' - \frac{\eta^2}{\alpha} P_{\mathrm{Ran}}$$

is self-adjoint on $\mathcal{D}(-\Delta)$ and bounded below.

Proof. $-\beta\Delta$ is self-adjoint on $\mathcal{D}(-\Delta)$, while $V_{q^\star}'' - (\eta^2/\alpha) P_{\mathrm{Ran}}$ is self-adjoint and either bounded on L^2 or relatively bounded with respect to $-\Delta$ with Kato bound < 1. Hence L_{eff} is self-adjoint on $\mathcal{D}(-\Delta)$ by the Kato–Rellich theorem [2]. □

Assumption 5.5 (Stable vacuum gate (no tachyonic directions))**.** We interpret $q^\star$ as a *stable vacuum* only in regimes where the effective open operator $L_{\text{eff}}(q^\star)$ is bounded below on the declared physical open subspace and has no negative-frequency directions, i.e. $\lambda^+_{\min}(L_{\text{eff}}(q^\star)) \geq 0$ (and in the massive channel $\lambda^+_{\min} > 0$). A sufficient criterion is the coercivity bound in Proposition 5.11 (e.g. $\beta\lambda_1 + \mu_0 > \eta^2/\alpha$ under Assumptions 5.2,5.3). When this gate fails, the stationary point is not treated as a vacuum but as an unstable/critical configuration.

Remark 5.14 (Negative spectrum and "tachyonic" language)*.* If $\inf \operatorname{spec}(L_{\text{eff}}(q^\star)|_{\mathcal{H}_R^{\text{phys}}}) < 0$, then the R3 linearization admits exponentially growing modes ($\omega^2 < 0$), which in QFT grammar may be described as "tachyonic". In this series this is interpreted as *instability of the stationary point*: such a $q^\star$ is not an R2-selected stable vacuum (Definition 5.1) and typically indicates proximity to a basin boundary/phase transition (Section 12.4).

9.2 Physical Hilbert subspace (constraint reduction)

Mass data are defined from $L_{\text{eff}}(q^\star)$ *after* applying the series-wide kernel/zero-mode conventions. We record them here as a single explicit definition, so that later references to "the mass spectrum" do not depend on prose.

Definition 5.3 (Physical subspace for spectral data)**.** Fix an operator instance (E_I, E_R, D_I) together with a declared convention for $\operatorname{Ker}(D_I)$ (series interface), and a background metric g. Let $\mathcal{H}_R := L^2(M; E_R)$ denote the open Hilbert space. Define the closed gauge reduction using $F := D_I I_I$ and working modulo $\operatorname{Ker}(D_I)$.

For the open sector, fix one of the following (series lock):

1. (**Mean-zero / gauge slice**) choose a closed subspace $\mathcal{H}_R^0 \subset \mathcal{H}_R$ such that a Poincaré-type inequality holds on $\mathcal{H}_R^0$ (e.g. mean-zero scalar slice, Coulomb gauge slice),
2. (**Mass-stabilized**) keep the full $\mathcal{H}_R$ but add a lower-order stabilization consistent with the contract.

By design, the choice of $\mathcal{H}_R^{\text{phys}}$ is a *declared slice/zero-mode convention* (a representation protocol datum), not an arbitrary modification of the primitive.

In either case, define the *physical open subspace* $\mathcal{H}_R^{\text{phys}}$ as the declared constraint subspace on which $L_{\text{eff}}(q^\star)$ is evaluated.

9.3 Spectral invariants: spectrum vs. gap

Definition 5.4 (Vacuum spectrum and mass gap)**.** Let $q^\star$ be an R2-selected vacuum (Definition 5.1). Consider $L_{\text{eff}}(q^\star)$ as a self-adjoint operator on $\mathcal{H}_R^{\text{phys}}$.

1. The *vacuum spectrum* is $\operatorname{spec}(L_{\text{eff}}(q^\star)|_{\mathcal{H}_R^{\text{phys}}})$.
2. The *vacuum mass gap* is the bottom of the *positive* spectrum:

$$\lambda^+_{\min}(q^\star) := \inf\big(\operatorname{spec}(L_{\text{eff}}(q^\star)|_{\mathcal{H}_R^{\text{phys}}}) \cap (0,\infty)\big), \qquad \omega_{\min}(q^\star) := \sqrt{\lambda^+_{\min}(q^\star)}. \tag{5.24}$$

Remark 5.15*.* The distinction between $\lambda_{\min}$ and $\lambda^+_{\min}$ matters when topological or gauge zero-modes are present. By definition, the mass gap is extracted after excluding forced zero eigenvalues.

9.4 de Rham instance: harmonic modes vs. massive channels

Under the de Rham instance (Instance 5.1), Hodge decomposition separates the open sector into exact/coexact/harmonic channels.

Corollary 5.14 (Harmonic zero-modes coexist with a positive mass gap)**.** *Assume Instance 5.1 and let*

$$u = u_{\text{ex}} + u_{\text{coex}} + u_{\text{harm}}$$

be the Hodge decomposition on $\mathcal{H}_R^{\text{phys}}$*. Then* $L_{\text{eff}}(q^\star)$ *decomposes as a block operator*

$$L_{\text{eff}}(q^\star) = \big(-\beta\Delta + V''_{q^\star}\big) \ \textit{on}\ (\text{coex} \oplus \text{harm}) \ \oplus\ \big(-\beta\Delta + V''_{q^\star} - \eta^2/\alpha\big) \ \textit{on}\ (\text{ex}),$$

*and in particular the Schur-induced order-*0 *term does not act on* harm*.*

If harm $\neq \{0\}$*, then* 0 *may belong to the spectrum due to topological modes. Nevertheless the mass gap is given by* (5.24) *and is controlled by the bottom of the* positive *spectrum on* (ex $\oplus$ coex) *(with the declared slice conventions).*

Proof. Under the de Rham instance, $\text{Ran}(D_I)$ coincides with the exact channel and the L^2 orthogonal decomposition is $\mathcal{H}_R^{\text{phys}} = \text{ex} \oplus \text{coex} \oplus \text{harm}$. Thus P_{Ran} acts as the projector onto ex and vanishes on coex $\oplus$ harm. Substituting $P_{\text{Ran}} = P_{\text{ex}}$ into (5.12) yields the stated block form. Possible harmonic zero-modes affect $0 \in \text{spec}$, but the mass gap is defined as the bottom of the *positive* spectrum (5.24). ☐

10 Dispersion (R3)

Part III fixes the conservative reading (R3) by minimal time completion. Linearizing the time-completed equation at the stationary background yields

$$\partial_t^2 u + L_{\text{eff}} u = 0, \tag{5.25}$$

where L_{eff} is (5.12).

Definition 5.5 (Variational mass spectrum)**.** Assume L_{eff} is self-adjoint and nonnegative on the declared subspace. Let $\lambda_j \in \text{spec}(L_{\text{eff}})$. Define $\omega_j := \sqrt{\lambda_j}$. A KG normalization instance converts ω_j to a mass parameter by a fixed convention (involving $c, \hbar$ in that instance).

Remark 5.16 (Compact vs. noncompact spectra)*.* The discrete eigenvalue notation is a convenience (e.g. compact M). On noncompact M, one reads the spectral measure/continuous spectrum of $L_{\text{eff}}(q^\star)$ on $\mathcal{H}_R^{\text{phys}}$, or uses a finite-mode/box-normalized instance (Section 12.3).

Remark 5.17 (Invariant vs. instance-dependent)*.* Invariant (within a fixed background and operator instance): the spectrum of L_{eff}. Instance-dependent: the numerical identification of λ with m^2 in standard units.

10.1 KG normalization instance (explicit conversion)

Write the effective operator as

$$L_{\text{eff}} = -\beta\Delta + M_{\text{eff}}^{(0)}, \qquad M_{\text{eff}}^{(0)} := V''_\star - \frac{\eta^2}{\alpha} P_{\text{Ran}}, \tag{5.26}$$

where $M_{\text{eff}}^{(0)}$ is order-0 (projector-valued in general). Then (5.25) is equivalently

$$\partial_t^2 u - \beta\Delta u + M_{\text{eff}}^{(0)} u = 0. \tag{5.27}$$

A convenient normalization is obtained by the time rescaling $\tilde{t} := \sqrt{\beta}\, t$ (a representation-level choice), yielding

$$\partial_{\tilde{t}}^2 u - \Delta u + \beta^{-1} M_{\text{eff}}^{(0)} u = 0. \tag{5.28}$$

In particular,

$$\beta^{-1} M_{\text{eff}}^{(0)} = \beta^{-1} V''_\star - \frac{\eta^2}{\alpha\beta} P_{\text{Ran}} = \beta^{-1} V''_\star - \kappa^2 P_{\text{Ran}}, \tag{5.29}$$

so the dimensionless spectral shift in the KG-normalized form is controlled by the invariant κ.

Remark 5.18 (Observation time vs. R3 time). The frequencies ω in (5.25)–(5.28) are defined with respect to the chosen R3 time parameter (and its normalization, e.g. $\tilde{t} = \sqrt{\beta}\, t$). If an readout instance uses a reparametrized clock $t_{\text{obs}} = f(\tilde{t})$ with $f' > 0$, then locally $\omega_{\text{obs}} = \omega/f'$, while the operator eigenvalues $\lambda = \omega^2 \in \operatorname{spec}(L_{\text{eff}})$ are unchanged. Thus the spectrum is the series-level invariant; the conversion to observed units is representation-level.

On any finite-mode truncation/QFT instance where $\beta^{-1} M_{\text{eff}}^{(0)}$ is represented by a matrix M_μ^2, one may identify (in the textbook Klein–Gordon normalization)

$$\left(\frac{mc}{\hbar}\right)^2 = \mathsf{M}_\mu^2, \tag{5.30}$$

with the understanding that the conversion constants $(c, \hbar)$ belong to that instance/normalization rather than to the series-level invariant. Experimental matching to reported particle masses requires fixing the observation/QFT instance and the validation rail; it is outside the scope of this structural paper.

In particular, in a local flat instance (plane-wave modes), (5.27) gives the dispersion relation

$$\omega^2 = \beta |k|^2 + \lambda^{(0)}, \tag{5.31}$$

where $\lambda^{(0)}$ is an eigenvalue of $M_{\text{eff}}^{(0)}$ on the chosen channel. Thus the *gap* at $k = 0$ is $\omega_0^2 = \lambda^{(0)}$, i.e. the order-0 Hessian data.

11 Higgs instance (factorization)

Scope warning (Higgs positioning).
No new structural claim is made here about electroweak symmetry breaking. Any "Higgs factorization" below is a *non-unique* representation-level parameterization of an already extracted mass matrix, used only as a comparison interface with standard QFT language.

This section makes explicit the separation: (i) the *mass operator data* are fixed by the backbone via the Hessian/Schur mechanism (Sections 5–10), whereas (ii) Higgs-language parameters are an *effective instance* used to re-express those data in a standard local QFT grammar.

11.1 Finite-mode instances: extracting mass matrices from operator data

Let $\mathcal{H}$ denote the declared open Hilbert space (after the series zero-mode convention), and let $V_N \subset \mathcal{H}$ be a finite-dimensional subspace (a "finite-mode instance") with basis $(\varphi_a)_{a=1}^N$. Define the Gram matrix

$$G_{ab} := \int_M \langle \varphi_a, \varphi_b \rangle \, \mathrm{dvol}_g, \tag{5.32}$$

the Laplace-principal operator (stiffness) matrix

$$K_{ab} := \beta \int_M \langle \nabla \varphi_a, \nabla \varphi_b \rangle \, \mathrm{dvol}_g, \tag{5.33}$$

and the extracted order-0 mass-squared matrix from the backbone data

$$(\mathsf{M}^2)_{ab} := \int_M \langle \varphi_a, (M_{\text{eff}}^{(0)} \varphi_b) \rangle \, \mathrm{dvol}_g, \qquad M_{\text{eff}}^{(0)} := V_\star'' - \frac{\eta^2}{\alpha} P_{\text{Ran}}. \tag{5.34}$$

Projecting the R3 linearized equation (5.27) to V_N yields the finite ODE system

$$G\, \ddot{q} + (K + \mathsf{M}^2)\, q = 0, \tag{5.35}$$

where $q(t) \in \mathbb{R}^N$ are the mode coefficients.

Proposition 5.15 (Extracted spectrum is basis-invariant)**.** *If $T \in GL(N)$ is a basis change $\tilde{\varphi} = \varphi T$, then $\tilde{G} = T^\top G T$, $\tilde{K} = T^\top K T$, and $\widetilde{\mathsf{M}}^2 = T^\top \mathsf{M}^2 T$. Hence the generalized eigenvalues of $(K + \mathsf{M}^2, G)$, i.e.*

$$(K + \mathsf{M}^2)a = \omega^2 \, Ga, \tag{5.36}$$

are invariant under basis changes. These ω^2 approximate $\mathrm{spec}(L_{\mathrm{eff}})$ *as V_N increases.*

Proof. Under a basis change $\tilde{\varphi} = \varphi T$, the coordinate vector transforms by $a = T^{-1}\tilde{a}$. The projected equation $G\ddot{q} + (K + \mathsf{M}^2)q = 0$ becomes $T^\top G T \, \ddot{\tilde{q}} + T^\top (K + \mathsf{M}^2) T \, \tilde{q} = 0$, giving the stated congruence transforms. The generalized eigenvalue equation (5.36) is therefore equivalent in the two bases, so the eigenvalues ω^2 are invariant. □

Remark 5.19 (What is fixed vs. chosen). The operator data $M^{(0)}_{\mathrm{eff}}$ are fixed by the backbone at the chosen stationary background. The finite-mode instance is a choice of representation (truncation/basis) of those fixed data.

11.2 Higgs parameterization as factorization

Once a local QFT instance is chosen, the order-0 data are represented by matrices (e.g. M^2 above). Higgs-language parameters can then be introduced as a factorization of the same matrix data.

Lemma 5.16 (Elementary factorization)**.** *Let M^2 be a Hermitian positive semidefinite matrix. For any $v > 0$, there exists a matrix Y such that*

$$\mathsf{M}^2 = \frac{v^2}{2} \, \mathsf{Y}^\dagger \mathsf{Y}. \tag{5.37}$$

In particular, for diagonal $\mathsf{M} = \mathrm{diag}(m_1, \ldots, m_N)$ one may set $y_i = \sqrt{2}\, m_i / v$.

Proof. Since $\mathsf{M}^2 \succeq 0$ is Hermitian, it has a unique positive semidefinite square root $(\mathsf{M}^2)^{1/2}$. Set $\mathsf{Y} := \sqrt{2}\, v^{-1} (\mathsf{M}^2)^{1/2}$. Then $\frac{v^2}{2} \, \mathsf{Y}^\dagger \mathsf{Y} = (\mathsf{M}^2)^{1/2} (\mathsf{M}^2)^{1/2} = \mathsf{M}^2$. □

Remark 5.20 (Positioning statement). Lemma 5.16 is linear algebra. Its role here is solely organizational: Higgs-language parameters (v, Y) are a representation-level factorization of already-derived mass data. They do not constitute a new primitive mass source at the fixed-density backbone level.

12 Phase structure and invariance

12.1 Mass as a map on vacua

Fix an R2-selected vacuum $q^\star$ (Definition 5.1). The effective open operator is evaluated at $q^\star$,

$$L_{\mathrm{eff}}(q^\star) = -\beta \Delta + V''_{q^\star} - \frac{\eta^2}{\alpha} \, P_{\mathrm{Ran}}, \tag{5.38}$$

and its order-0 sector

$$M^{(0)}_{\mathrm{eff}}(q^\star) := V''_{q^\star} - \frac{\eta^2}{\alpha} \, P_{\mathrm{Ran}} \tag{5.39}$$

is the *mass data* in the series sense.

Definition 5.6 (Mass spectrum attached to a vacuum)**.** Fix a vacuum $q^\star$ (Definition 5.1) and the physical open subspace $\mathcal{H}_R^{\mathrm{phys}}$ (Definition 5.3). The *vacuum mass spectrum* is the spectrum

$$\mathrm{spec}\big(L_{\mathrm{eff}}(q^\star)|_{\mathcal{H}_R^{\mathrm{phys}}}\big).$$

Equivalently, in the R3 reading it is the dispersion spectrum $\omega^2 \in \mathrm{spec}(L_{\mathrm{eff}}(q^\star)|_{\mathcal{H}_R^{\mathrm{phys}}})$ of (5.25).

12.2 Basin-constancy (phase rigidity inside a vacuum)

Let $\mathcal{B}(q^\star)$ denote the basin of attraction of $q^\star$ under the R2 flow (5.4). Define the selection map (when it exists) by $\mathcal{V}(q_0) := \lim_{\tau\to\infty} q(\tau; q_0)$.

Proposition 5.17 (Mass data are constant on a basin)**.** *Assume that for all $q_0 \in \mathcal{B}(q^\star)$ the R2 flow converges to the same limit $q^\star$. Then for any $q_0, q_0' \in \mathcal{B}(q^\star)$,*

$$L_{\mathrm{eff}}(\mathcal{V}(q_0)) = L_{\mathrm{eff}}(\mathcal{V}(q_0')) = L_{\mathrm{eff}}(q^\star),$$

and in particular the vacuum mass spectrum (Definition 5.6) is identical.

Proof. By assumption, $\mathcal{V}(q_0) = \mathcal{V}(q_0') = q^\star$ for all initial data in the basin. Evaluating L_{eff} at the limit gives the same operator, hence the same spectrum, for all q_0, q_0' in $\mathcal{B}(q^\star)$. □

Remark 5.21 (What changes masses). Within a fixed basin, nothing changes: $q^\star$ is fixed, hence so is $D^2E(q^\star)$ and its Schur-complement spectrum. Mass changes only when the limiting vacuum changes, i.e. when one crosses a basin boundary (phase boundary).

12.3 Finite-mode instance invariance

In a finite-mode (QFT/observation) instance, choose a basis $\Phi = \{\varphi_a\}_{a=1}^N$ of a subspace $V_N \subset \Gamma(E_R)$. Define the matrices

$$G_{ab} := \int_M \langle \varphi_a, \varphi_b\rangle \,\mathrm{dvol}_g, \tag{5.40}$$

$$K_{ab} := \beta \int_M \langle \nabla\varphi_a, \nabla\varphi_b\rangle \,\mathrm{dvol}_g, \tag{5.41}$$

$$(M^2)_{ab} := \int_M \langle \varphi_a, (M^{(0)}_{\mathrm{eff}}(q^\star)\,\varphi_b)\rangle \,\mathrm{dvol}_g, \tag{5.42}$$

so that the Galerkin projection of (5.25) yields

$$G\,\ddot{q} + (K + M^2)\,q = 0, \qquad (K+M^2)a = \omega^2\,Ga. \tag{5.43}$$

Lemma 5.18 (Generalized eigenvalues are basis-invariant)**.** *Let $\tilde\Phi = \{\tilde\varphi_a\}$ be another basis of V_N with $\tilde\varphi = \varphi S$ for an invertible matrix S. Then*

$$\tilde G = S^\dagger G S, \quad \tilde K = S^\dagger K S, \quad \widetilde{M^2} = S^\dagger (M^2) S,$$

and the generalized eigenvalues ω^2 of $(K+M^2)a = \omega^2 Ga$ coincide with those of $(\tilde K + \widetilde{M^2})\tilde a = \omega^2 \tilde G \tilde a$.

Proof. Under the basis change $\tilde\varphi = \varphi S$ one has $\tilde G = S^\dagger G S$ and $\tilde K + \widetilde{M^2} = S^\dagger (K+M^2) S$, so the pencil $(K+M^2,\ G)$ is congruent to $(\tilde K + \widetilde{M^2},\ \tilde G)$ and the generalized eigenvalues coincide. □

12.4 Phase boundaries and gap closing (regime statement)

Let $\lambda^+_{\min}(q^\star)$ denote the bottom of the spectrum of $L_{\mathrm{eff}}(q^\star)$ on the declared constraint subspace (after kernel/zero-mode conventions). Define the *gap*

$$\omega_{\min}(q^\star) := \sqrt{\lambda^+_{\min}(q^\star)}. \tag{5.44}$$

Remark 5.22 (Generic transition signature). If a basin boundary is approached where the R2 limit ceases to be unique (or where stability is lost), the Hessian typically develops a null direction on the constraint subspace, i.e. $\lambda^{+}_{\min}(q^{\star}) \to 0$. In the R3 reading this is the closing of the dispersion gap $\omega_{\min} \to 0$. This provides a structural notion of "phase transition": masses change only through gap-closing events.

13 Mass extraction protocol

This section records a minimal pipeline that turns the fixed density $e(\cdot)$ into a reproducible mass spectrum, using only the series-locked operations and the declared operator/readout instances.

1. **Choose an operator instance.** Fix $(E_I, E_R, D_I, \mathrm{Ker}(D_I)$ treatment) and the physical open subspace $\mathcal{H}_R^{\mathrm{phys}}$ (Definition 5.3). This fixes P_{Ran} and the active channels.

2. **Select a vacuum by R2.** Pick an initial condition q_0 and run the R2 flow (5.4). Within a basin, $q(\tau) \to q^{\star}$ (Theorem 5.2) and the mass data are evaluated at $q^{\star}$.

3. **Form the Hessian blocks.** Compute the quadratic form (5.8) at $q^{\star}$ and identify the block operators (A, B, C) in (5.9). In the canonical representative,

$$A = -\beta\Delta + V''_{q^{\star}}, \qquad C = \alpha I, \qquad B = \eta P_{\mathrm{Ran}}.$$

4. **Eliminate the closed variable (Schur complement).** Define the effective open operator

$$L_{\mathrm{eff}}(q^{\star}) = A - BC^{-1}B^{*}$$

   (Theorem 5.3), i.e. (5.12). Its order-0 sector is $M^{(0)}_{\mathrm{eff}}(q^{\star})$ (5.39).

5. **Read the spectrum (R3).** The R3 linearization is $\partial_t^2 u + L_{\mathrm{eff}}(q^{\star})u = 0$ (5.25). The vacuum spectrum and gap are the spectral invariants in Definition 5.4.

6. **Optional finite-mode (QFT/observation) instance.** Choose a finite subspace V_N and basis Φ and compute (G, K, M^2) by (5.40)–(5.42). Solve the generalized eigenproblem (5.43). The resulting ω^2 are basis-invariant (Lemma 5.18) and depend only on the chosen subspace V_N.

7. **Optional Higgs instance.** If one wishes to represent the same finite-mode mass data in local SM grammar, factorize M^2 as $\mathsf{M}^2 = (v^2/2)\mathsf{Y}^{\dagger}\mathsf{Y}$ (Lemma 5.16).

Remark 5.23 (What is fixed vs. what is chosen). Fixed by the series backbone: the density $e(\cdot)$ and the operations R1/R2/R3. Chosen by instances: the operator instance (hence P_{Ran}), the slice conventions (hence $\mathcal{H}_R^{\mathrm{phys}}$), and the observation/QFT truncation subspace V_N.

14 Failure map (diagnostic branching)

Because Part V reorganizes the *status* of mass (input $\to$ variational spectral data), it is useful to record a minimal "failure map": if a target interpretation conflicts with the forced Schur structure, the table below indicates which part of the series contract is being tested.

Target requirement / interpretation	What must fail (and what to revise)
Need a momentum-dependent "mass" that alters the principal symbol (e.g. UV dispersion exponents not inherited from the Laplace principal operator)	The backbone/operator-class lock is being violated: one must abandon the Laplace-type principal operator contract (higher-order/fractional kinetic) or accept that the effect is not mediated by a static Schur elimination.
Need a mass shift on modes with $P_{\mathrm{Ran}}u = 0$ (channels declared gauge/non-observable by the operator instance)	The declared operator instance (hence P_{Ran} and channel typing) is inconsistent with the intended interpretation; revise (E_I, E_R, D_I) or change the declared physical channel.
Phenomenology demands independent dimensionless couplings beyond the invariant $\kappa = \eta/\sqrt{\alpha\beta}$	The single-invariant claim is being tested: introduce additional closed blocks/invariants or enlarge the equivalence class (to admit additional independent couplings).
A supposed "vacuum" exhibits tachyonic directions ($\omega^2 < 0$) but is still interpreted as stable	The admissible-vacuum gate is failing: reinterpret the configuration as a basin boundary/phase transition signature (Section 12.4) or select a different R2 limit as the vacuum.
Joint analysis with the dual-Schur/closed rail indicates open/closed critical mismatch	At least one hypothesis of the critical equivalence theorem (Theorem 5.4) is not satisfied on the chosen instance: the comparison is not being made at the same declared vacuum/instance, A is not invertible on the declared slice, or the effect involves additional dynamics beyond static elimination.

Remark 5.24. The failure map is not a patch list. It is a logic map: it states which *gate/contract* is being tested when the forced Schur-induced mass structure is confronted with a target interpretation.

15 Summary

1. **Vacuum selection is internal (R2⇒R1).** The R2 gradient flow (5.4) dissipates E_{fld} (5.5). Under the analytic-gradient regime (Assumption 5.1) and precompactness, it converges to a single stationary limit $q^\star$ (Theorem 5.2), which serves as the vacuum (Definition 5.1).

2. **Mass is order-0 Hessian spectral data at $q^\star$.** With the Laplace-type principal part locked, mass can only enter through order-0 curvature data of the R1 Hessian at $q^\star$, after applying the physical subspace convention (Definition 5.3).

3. **Closed elimination forces a Schur term (projector-valued).** With a coercive closed block ($C > 0$) and nonzero coupling, eliminating $F := D_I I_I$ yields the Schur complement $H_{\mathrm{eff}} = A - BC^{-1}B^*$ (Theorem 5.3). In the canonical representative this is $L_{\mathrm{eff}} = -\beta\Delta + V''_{q^\star} - (\eta^2/\alpha)P_{\mathrm{Ran}}$ (Proposition 5.6).

4. **Spectrum and gap are defined after zero-mode removal.** The vacuum spectrum is $\mathrm{spec}(L_{\mathrm{eff}}(q^\star)|_{\mathcal{H}_R^{\mathrm{phys}}})$ and the mass gap is the bottom of the positive spectrum $\lambda^+_{\min}(q^\star)$ (Definition 5.4).

5. **de Rham instance: exact-only action and topological zero-modes.** For $D_I = d$, $P_{\mathrm{Ran}} = P_{\mathrm{ex}} = d\Delta_0^{-1}d^*$ and the Schur shift acts only on the exact channel (Theorem 5.12).

Harmonic modes are topological and may produce forced zero eigenvalues, while the mass gap remains a positive-spectrum invariant (Corollary 5.14).

6. **Finite-mode instances extract basis-invariant mass spectra.** A truncation V_N yields $(K + M^2)a = \omega^2 Ga$ (5.43) with M^2 extracted from $M^{(0)}_{\text{eff}}(q^\star)$. The generalized eigenvalues are basis-invariant (Lemma 5.18).
7. **Phase structure: masses change only by changing vacua.** Within a basin $\mathcal{B}(q^\star)$ the limit $q^\star$ is fixed, hence the mass spectrum is constant (Proposition 5.17). Mass changes only across basin boundaries, typically through gap-closing events (Section 12.4).
8. **Higgs is a representation-level factorization.** Higgs-language parameters (v, Y) realize a factorization $\mathsf{M}^2 = (v^2/2)\mathsf{Y}^\dagger\mathsf{Y}$ (Lemma 5.16) of already-derived mass data.

Next parts (series plan). Part VI develops the dual-Schur closed-curvature structural regime. Part VII is a pre-rail synthesis paper that packages elimination duality, no-escape shape-class rigidity, criticality equivalence, and order-independent structural invariants. Validation rails (cosmology/lensing/data comparison) are deferred to a post-Part XII rail dossier (or a separate rail paper).

Main statements (one-page box)

(S1) Vacuum selection. The R2 reading $\partial_\tau q = -\nabla E_{\text{fld}}(q)$ dissipates energy (5.5) and, under the analytic-gradient regime (Assumption 5.1) and precompactness, converges to a single stationary limit $q^\star$ (Theorem 5.2). This $q^\star$ is the vacuum (Definition 5.1).

(S2) Mass operator (forced Schur term). At $q^\star$ the effective open Hessian is the Schur complement $H_{\text{eff}} = A - BC^{-1}B^*$ (Theorem 5.3). In the canonical representative,

$$L_{\text{eff}}(q^\star) = -\beta\Delta + V''_{q^\star} - \frac{\eta^2}{\alpha}P_{\text{Ran}}, \qquad M^{(0)}_{\text{eff}}(q^\star) = V''_{q^\star} - \frac{\eta^2}{\alpha}P_{\text{Ran}}.$$

The vacuum spectrum and mass gap are defined on $\mathcal{H}_R^{\text{phys}}$ (Definitions 5.3–5.4).

(S3) Invariance and phase structure. Within an R2 basin $\mathcal{B}(q^\star)$ the limit $q^\star$ is fixed, hence the mass spectrum is basin-constant (Proposition 5.17); masses change only across basin boundaries, typically through gap closing (Section 12.4). Finite-mode instances extract basis-invariant spectra via the generalized eigenproblem (5.43) (Lemma 5.18). Higgs language is a representation-level factorization of the resulting mass matrix (Lemma 5.16).

A Electroweak Higgs instance (compatibility statement)

This appendix is a dictionary between the mass data extracted in this paper and standard electroweak (Higgs-sector) notation. It makes no derivation claim for the gauge group, couplings, or Higgs potential; it only fixes a conventional local QFT instance in which the same mass matrices are represented (cf. [7]).

A.1 Instance scope

Fix a finite-mode instance $V_N = \text{span}\{\varphi_a\}$ and the extracted order-0 mass-squared matrix M^2 from (5.42). In the electroweak grammar, fermion masses are parametrized by Yukawa matrices and a scale v, and gauge-boson masses by the standard covariant-derivative mass terms after symmetry breaking.

A.2 Fermion sector: Yukawa factorization

Given any Hermitian positive semidefinite mass-squared matrix M^2 on a finite set of species/modes, fix any $v > 0$. Then there exists a Yukawa matrix Y such that

$$\mathsf{M}^2 = \frac{v^2}{2}\,\mathsf{Y}^\dagger\mathsf{Y}. \tag{5.45}$$

This is exactly the factorization recorded in Lemma 5.16. In the SM grammar, one writes a Yukawa interaction and, after choosing a vacuum expectation value $\langle\Phi\rangle = (0,\ v/\sqrt{2})^\mathsf{T}$, obtains mass matrices $m = \mathsf{Y}v/\sqrt{2}$. Here the direction of implication is reversed: the mass data are fixed first by $L_{\text{eff}}(q^\star)$, and (5.45) provides a coordinate expression.

A.3 Gauge sector: W/Z mass dictionary

In the electroweak instance, choose gauge couplings (g, g') and define the weak mixing angle by

$$\tan\theta_W := \frac{g'}{g}, \qquad \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}. \tag{5.46}$$

After symmetry breaking at scale v, the standard mass relations are

$$m_W = \frac{gv}{2}, \qquad m_Z = \frac{\sqrt{g^2 + g'^2}}{2}\,v, \qquad m_W = m_Z\cos\theta_W. \tag{5.47}$$

In this paper the gauge sector is not derived. If a chosen local QFT instance identifies vector-boson modes with eigenvalues interpreted as m_W^2 and m_Z^2, one may choose (v, g, g') to satisfy (5.47); equivalently, for fixed $v > 0$ set $g := 2m_W/v$ and $g' := \sqrt{(2m_Z/v)^2 - g^2}$ (when $m_Z \geq m_W$). This step is a reparametrization of identified eigenvalues into standard electroweak notation and carries no prediction at the backbone level.

A.4 Higgs scalar: positioning

In electroweak notation the Higgs mass is the curvature of a scalar potential at the vacuum. Here it is treated as additional order-0 spectral data within a chosen instance. This paper does not prescribe the Higgs potential and makes no Higgs-mass prediction; it only fixes the positioning: Higgs-sector parameters provide a conventional coordinate system for mass matrices already fixed by $L_{\text{eff}}(q^\star)$.

B Worked example: flat torus and longitudinal mass shift

Example on $M = \mathbb{T}^n$ (flat) showing that P_{ex} is the longitudinal projector and the Schur term shifts only longitudinal modes.

B.1 Setup

Let $M = \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$ with the flat metric. Use the de Rham instance with $k = 0$:

$$E_I = \Lambda^0 T^*M, \qquad E_R = \Lambda^1 T^*M, \qquad D_I = d, \qquad F = dI_I \in \Omega^1(M).$$

Thus the open variable is a 1-form $I_R \in \Omega^1(M)$ and $P_{\text{Ran}} = P_{\text{ex}} = d\Delta_0^{-1}d^*$. Assume for simplicity that $V''_{q^\star} = m_{\text{open}}^2\,\text{Id}$ is a constant zeroth-order endomorphism on $\Omega^1(M)$. Here $m_{\text{open}} > 0$ is a *vacuum curvature scale on the open slice* (a constant order-0 term in the open operator); it should not be confused with any optional regulator parameter m_0 used elsewhere purely for zero-mode/coercivity control.

B.2 Fourier decomposition and the exact projector

Let $k \in \mathbb{Z}^n$ and consider a Fourier mode $u(x) = a\, e^{ik\cdot x}$ where $a \in \mathbb{C}^n$ is a constant coefficient (identified with a 1-form). For $k \neq 0$ one has

$$d^*u = i\, k \cdot a\, e^{ik\cdot x}, \qquad \Delta_0^{-1}(e^{ik\cdot x}) = -\frac{1}{|k|^2}\, e^{ik\cdot x},$$

(using the sign convention $\Delta e^{ik\cdot x} = -|k|^2 e^{ik\cdot x}$), hence

$$P_{\mathrm{ex}}u = d\Delta_0^{-1}d^*u = \frac{k(k\cdot a)}{|k|^2}\, e^{ik\cdot x}. \tag{5.48}$$

Therefore P_{ex} is the *longitudinal projector*: it extracts the component of a parallel to k. Write $a = a_{\parallel} + a_{\perp}$ with $a_{\parallel} \parallel k$ and $a_{\perp} \cdot k = 0$; then

$$P_{\mathrm{ex}}u = a_{\parallel}e^{ik\cdot x}, \qquad (I - P_{\mathrm{ex}})u = a_{\perp}e^{ik\cdot x}.$$

Exact/coexact identification. For $k \neq 0$, the scalar mode $\phi(x) = e^{ik\cdot x}$ satisfies $d\phi = i\, k\, e^{ik\cdot x}$, so a 1-form Fourier mode $u(x) = a\, e^{ik\cdot x}$ is exact iff $a \parallel k$. Moreover $d^*u = 0$ iff $k \cdot a = 0$, i.e. a is transverse; on the flat torus this is the coexact subspace at frequency k. Thus P_{ex} is precisely the Hodge projector onto exact 1-forms for each $k \neq 0$. In particular, for each $k \neq 0$ there is one shifted longitudinal (exact) polarization and $(n-1)$ unshifted transverse (coexact) polarizations.

B.3 Shifted eigenvalues and the κ-invariant coefficient

For $k \neq 0$, the Hodge Laplacian on 1-forms satisfies $-\Delta u = |k|^2 u$. Thus the effective operator

$$L_{\mathrm{eff}} = -\beta\Delta + m_{\mathrm{open}}^2\,\mathrm{Id} - \frac{\eta^2}{\alpha}P_{\mathrm{ex}}$$

acts diagonally on the longitudinal/transverse splitting:

$$L_{\mathrm{eff}}(a_{\perp}e^{ik\cdot x}) = (\beta|k|^2 + m_{\mathrm{open}}^2)\, a_{\perp}e^{ik\cdot x}, \tag{5.49}$$

$$L_{\mathrm{eff}}(a_{\parallel}e^{ik\cdot x}) = (\beta|k|^2 + m_{\mathrm{open}}^2 - \eta^2/\alpha)\, a_{\parallel}e^{ik\cdot x}. \tag{5.50}$$

After KG normalization $\tilde{t} = \sqrt{\beta}\, t$ one reads

$$\omega_{\perp}^2 = |k|^2 + \frac{m_{\mathrm{open}}^2}{\beta}, \qquad \omega_{\parallel}^2 = |k|^2 + \frac{m_{\mathrm{open}}^2}{\beta} - \frac{\eta^2}{\alpha\beta} = |k|^2 + \frac{m_{\mathrm{open}}^2}{\beta} - \kappa^2,$$

exhibiting explicitly the invariant coefficient $\eta^2/(\alpha\beta) = \kappa^2$ (cf. (5.2) and (5.29)).

Order-0 nature. The shift $-\eta^2/\alpha$ (equivalently $-\kappa^2$ in KG-normalized units) is independent of k, confirming that elimination generates a zeroth-order mass term and does not alter the Laplace-type principal operator.

If $\kappa^2 > m_{\mathrm{open}}^2/\beta$, then $\omega_{\parallel}^2 < 0$ for sufficiently small $|k|$, indicating an unstable stationary point (cf. Section 12.4).

B.4 Zero modes

For $k = 0$, u is harmonic, $d^*u = 0$, and $P_{\mathrm{ex}}u = 0$. This is the simplest instance of Corollary 5.14.

Glossary (series meta-language)

- **Structural Layer**: the fixed backbone (primitive E plus admissible equivalences) and the admissible readings (R1/R2/R3).
- **Reading (R1/R2/R3)**: operations applied to the same backbone: R1 (stationarity/EL), R2 (gradient/dissipative), R3 (minimal time-completion/conservative).
- **Elimination**: a representation-level representative selection step (e.g. Schur/dual-Schur reduction) that preserves the declared principal symbol / principal operator lock.
- **Representation instance**: a concrete protocol choice specifying a readout/retention map and base measure.
- **Representation Layer**: protocol data, typically $(\Pi, \mu, \mathcal{G}, \eta)$ where Π is a readout map, μ a base measure, and $(\mathcal{G}, \eta)$ update/event/summary rules.
- **Notation lock ($\mathcal{G}$ vs. G)**: G is reserved for structural variables/operators; $\mathcal{G}$ denotes protocol/update rules in instances.
- **Gate / regime gate**: an explicit validity condition (bounds, coercivity, smallness, slow-variation, etc.) delimiting the regime where a closure statement is licensed.
- **Stopping rule**: an explicit declared exit condition (when a regime/rail no longer applies).
- **Rail**: an optional validation pipeline (pass/fail + diagnostics) attached after fixing an instance; not part of the structural theorem statements.
- **Time instance alias**: a declared synchronization/parameterization protocol used to interpret time variables across regimes.

Declarations

Data availability. No new data were generated or analyzed in this study.

Code availability. No code was used to generate results in this manuscript.

Competing interests. The author declares no competing interests.

Funding. This research received no external funding.

Scope. All statements are sectorial/regime-level within the declared regime and admissibility gates, and are subject to explicit stopping rules where readout instances are used; no global continuation beyond regime exit is claimed.

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