Instabilizability Conditions for Continuous-Time Stochastic Systems Under Control Input Constraints

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Abstract—**In this paper, we investigate constrained control of continuous-time linear stochastic systems. We show that for certain system parameter settings, constrained control policies can never achieve stabilization. Specifically, we explore a class of control policies that are constrained to have a bounded average second moment for Ito-type stochastic differential equations with additive and multiplicative noise. We prove that in certain settings of the system parameters and the bounding constant of the control constraint, divergence of the second moment of the system state is inevitable regardless of the initial state value and regardless of how the control policy is designed.**

Index Terms—**Stochastic systems, constrained control, linear systems**

I. INTRODUCTION

STABILIZATION under control input constraints is an important research problem due to its wide applicability to systems with actuator saturation. The works [1], [2] describe the challenges of this problem and provide comprehensive discussions of the important results. A key result on this problem is an impossibility result: linear deterministic systems with strictly unstable system matrices cannot be globally stabilized if the norm of the control input is constrained to stay below a constant threshold [3], [4].

There is a rapidly growing interest in exploring control input constraints for stochastic systems. For instance, [5]–[8] proposed stochastic model predictive controllers with control constraints; [9] and [10] developed reinforcement learning control frameworks with constraints, [11] investigated fuzzy controllers for stochastic systems with actuator saturation. Constrained control of nonlinear stochastic systems was investigated by [12] and [13], and moreover, [14] explored control constraints in stochastic networked control systems.

The work [15] presented an impossibility result for constrained control of discrete-time stochastic systems. It was shown there that if the control input of a strictly unstable discrete-time stochastic system is subject to hard norm-constraints, then the second moment of the state always diverges under nonvanishing and unbounded additive stochastic process noise. A common approach to overcome the difficulties in the stabilization of strictly unstable systems is to consider probabilistic constraints instead of hard deterministic constraints. However, it was shown in [16] that under certain conditions, stabilization of a discrete-time linear stochastic system is impossible even under probabilistic constraints.

The scope of the impossibility results provided in the above-mentioned articles covers discrete-time stochastic systems with additive noise. In this paper, we are motivated to expand this scope by addressing two issues. First, we want to know if similar impossibility results can be obtained for *continuous-time* stochastic systems. Secondly, we want to investigate the effects of both *additive* and *multiplicative* noise terms. Handling multiplicative noise terms is important, since such terms can characterize parametric uncertainties in the system (see [17], [18]). As our main contribution, we identify the scenarios where stabilization of a *continuous-time stochastic system (with both additive and multiplicative noise)* is *not possible* under *probabilistically-constrained control policies*. Specifically, we consider control policies that have bounded time-averaged second moments. This class of control policies encapsulate many types of controllers with (probabilistic or deterministic) control constraints. We obtain conditions on the bounding value of the control constraint, under which the second moment of the state *diverges* regardless of the controller choice and regardless of the initial state value.

Our analysis for the continuous-time systems with additive and multiplicative noise has a few key differences from that for the discrete-time case with additive-only noise provided in [15], [16]. First, in our case, we handle Ito-type stochastic differential equations with state-dependent noise terms characterizing multiplicative Wiener noise. In addition, we develop a form of reverse Gronwall's inequality to obtain lower bounds on functions with superlinear growth. Through our analysis, we observe that combination of additive and multiplicative noise can make systems harder to stabilize. Even systems that have Hurwitz-stable system matrices can be impossible to stabilize with constrained controllers under the combination of additive and multiplicative noise.

We organize the rest of the paper as follows. In Section II, we describe the constrained control problem. Then in Sections III and IV, we provide our impossibility results for constrained control of continuous-time stochastic systems. Finally, we present numerical examples in Section V and conclude the paper in Section VI.

Notation: We denote the Euclidean norm by $\|\cdot\|$, the trace operator by $\mathrm{tr}(\cdot)$, and the maximum eigenvalue of a Hermitian matrix $H \in \mathbb{C}^{n \times n}$ by $\lambda_{\max}(H)$. We use $H^{\frac{1}{2}}$ to represent the unique nonnegative-definite Hermitian square root of a nonnegative-definite Hermitian matrix $H \in \mathbb{C}^{n \times n}$, satisfying $H^{\frac{1}{2}} H^{\frac{1}{2}} = H$ and $(H^{\frac{1}{2}})^* = H^{\frac{1}{2}}$. The identity matrix in $\mathbb{R}^{n \times n}$ is denoted by I_n. The notations $\mathbb{P}[\cdot]$ and $\mathbb{E}[\cdot]$ respectively denote the probability and expectation on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with sample space Ω and σ-algebra $\mathcal{F}$. We consider a continuous-time filtration $\{\mathcal{F}_t\}_{t \geq 0}$ with $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}$ for $t_1 \leq t_2$. Throughout the paper $\{W(t) = [W_1(t), \ldots, W_\ell(t)]^{\mathrm{T}} \in \mathbb{R}^\ell\}_{t \geq 0}$ denotes the Wiener process. Here, for every $i \in \{1, \ldots, \ell\}$, the process $\{W_i(t) \in \mathbb{R}\}_{t \geq 0}$ is $\mathcal{F}_t$-adapted; $\{W_i(t) \in \mathbb{R}\}_{t \geq 0}$, $i \in \{1, \ldots, \ell\}$, are independent processes. Moreover, $\overline{c}$ denotes the complex conjugate of a complex number $c \in \mathbb{C}$, and $\mathrm{Re}(c)$ denotes its real part. We use C^* to denote the complex conjugate transpose of a complex matrix $C \in \mathbb{C}^{n \times m}$, that is, $(C^*)_{i,j} = \overline{C_{j,i}}$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. Given a vector $v \in \mathbb{R}^n$, and indices $i, j \in \{1, \ldots, n\}$, $i \leq j$, we define $v_{i:j} \in \mathbb{R}^{j-i+1}$ as $v_{i:j} \triangleq [v_i, \ldots, v_j]^{\mathrm{T}}$.

II. CONSTRAINED CONTROL OF CONTINUOUS-TIME LINEAR STOCHASTIC SYSTEMS

Consider the continuous-time linear stochastic system described by the Ito-type stochastic differential equation

$$\mathrm{d}x(t) = (Ax(t) + Bu(t))\mathrm{d}t + [\Psi(x(t)), D]\,\mathrm{d}W(t), \quad (1)$$

for $t \geq 0$, where $x(t) \in \mathbb{R}^n$ is the state with deterministic initial value $x(0) = x_0$, $u(t) \in \mathbb{R}^m$ is the control input, and moreover, $\{W(t) \in \mathbb{R}^\ell\}_{t \geq 0}$ is the Wiener process.

The matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are called system and input matrices, respectively. Moreover, $\Psi(x(t)) \in \mathbb{R}^{n \times \ell_1}$ and $D \in \mathbb{R}^{n \times \ell_2}$ (with $\ell_1 + \ell_2 = \ell$) are noise matrices. The matrix-valued function $\Psi \colon \mathbb{R}^n \to \mathbb{R}^{n \times \ell_1}$ characterizes the effects of multiplicative noise and it is given by

$$\Psi(x) = [C_1 x, C_2 x, \ldots, C_{\ell_1} x], \quad (2)$$

where $C_i \in \mathbb{R}^{n \times n}$, $i \in \{1, \ldots, \ell_1\}$. The matrix $D \in \mathbb{R}^{n \times \ell_2}$ in (1) is used for characterizing the effects of additive noise.

Notice that $W_{1:\ell_1}(\cdot)$ enters in the dynamics as multiplicative noise and $W_{\ell_1+1:\ell}(\cdot)$ enters as additive noise, since $[\Psi(x(t)), D]\,\mathrm{d}W(t) = \Psi(x(t))\mathrm{d}W_{1:\ell_1}(t) + D\mathrm{d}W_{\ell_1+1:\ell}(t)$.

In this paper, we are interested in a stabilization problem. Since the Wiener process $W_{\ell_1+1:\ell}(\cdot)$ enters in the dynamics in an additive way, the state and its moments cannot converge to 0 regardless of the control input, unless $D = 0$. For this reason, asymptotic stabilization is not possible and a weaker notion of stabilization is needed. In this paper, we consider the *bounded second-moment stabilization* notion, where the control goal is to achieve $\sup_{t \geq 0} \mathbb{E}[\|x(t)\|^2] < \infty$.

We consider a stochastic constraint such that the time-averaged 2nd moment of $u(t)$ is bounded by $\hat{u} \geq 0$, i.e.,

$$\frac{1}{t} \int_0^t \mathbb{E}[\|u(\tau)\|^2]\mathrm{d}\tau \leq \hat{u}, \quad t \geq 0. \quad (3)$$

This constraint is a relaxation of other types of control constraints, i.e., the satisfaction of (3) does not necessarily imply

satisfaction of other constraints. Note on the other hand that norm-constraints (e.g., $\|u(t)\| \leq \overline{u}$ or $\|u(t)\|_\infty \leq \overline{u}$), time-averaged norm constraints (e.g., $\frac{1}{t} \int_0^t \|u(\tau)\|\mathrm{d}\tau \leq \overline{u}$), as well as first- and second-moment constraints (e.g., $\mathbb{E}[\|u(t)\|] \leq \overline{u}$ or $\mathbb{E}[\|u(t)\|^2] \leq \overline{u}$) all satisfy (3) for certain values of $\hat{u}$.

Remark 2.1: The structure of (3) is motivated by the networked control problem of a plant with a remotely located controller. In this problem, control commands $u_{\mathrm{C}}(t)$ transmitted from the controller are subject to packet losses, and the plant sets its input $u(t)$ to 0 if there is a packet loss (and to $u_{\mathrm{C}}(t)$ otherwise). The actuator at the plant side has a hard constraint requiring $\|u_{\mathrm{C}}(t)\|^2 \leq \overline{u}_{\mathrm{C}}$ for $t \geq 0$. With randomness involved in packet losses, the plant input $u(t)$ actually satisfies (3) with $\hat{u} < \overline{u}_{\mathrm{C}}$ (see Section IV.D of [16] for the specific form of $\hat{u}$). Even though the actuator may be powerful ($\overline{u}_{\mathrm{C}}$ is large), unstable noisy plants in certain scenarios cannot be stabilized if there are very frequent packet losses, because in such cases $\hat{u}$ is much smaller than $\overline{u}_{\mathrm{C}}$, and the controller is unable to provide inputs with sufficient average energy to the plant. ◁

For given A, B, Ψ, D, our goal is to find a threshold for $\hat{u}$, below which stabilization of (1) becomes impossible and the second moment $\mathbb{E}[\|x(t)\|^2]$ diverges regardless of the controller design.

The following lemmas are used in the derivation of our main result in Section III. The first lemma is related to the bounding value $\hat{u}$ of the control constraint (3). The second lemma is an extension of Gronwall's lemma (see, e.g., [19]), where the key condition involves a linear term and the result provides a lower bound instead of an upper bound.

Lemma 2.2: Let $\hat{u} \in [0, \infty)$, $\kappa \in (0, \infty)$ be scalars that satisfy $\hat{u} < \kappa$. Then $\mathcal{Q} \triangleq \{q > 1 \colon \hat{u} < \kappa/q\}$ is non-empty.

Proof: Let $\tilde{q} \triangleq 2\kappa/(\hat{u} + \kappa)$. Since $\hat{u} < \kappa$, we have $2\kappa > \hat{u} + \kappa$, which implies $\tilde{q} > 1$. Moreover, since $\hat{u} < \kappa$, we have $\kappa/\tilde{q} = (\hat{u}+\kappa)/2 > \hat{u}$. As both $\tilde{q} > 1$ and $\hat{u} < \kappa/\tilde{q}$ hold, we have $\tilde{q} \in Q$, implying that $\mathcal{Q} \neq \emptyset$. □

Lemma 2.3: Given scalars $c_0, c_1 \in \mathbb{R}$ and $\phi > 0$, suppose that $h \colon [0, \infty) \to \mathbb{R}$ is a continuous function that satisfies

$$h(t) \geq c_0 + c_1 t + \phi \int_0^t h(\tau)\mathrm{d}\tau, \quad t \geq 0. \quad (4)$$

Then we have

$$h(t) \geq c_0 e^{\phi t} + (c_1/\phi)(e^{\phi t} - 1), \quad t \geq 0. \quad (5)$$

Moreover, if (4) holds with equality, then (5) holds with equality.

Proof: Let $g(s) \triangleq c_0 + c_1 s + \phi \int_0^s h(\tau)\mathrm{d}\tau$ for $s \geq 0$. Since h is a continuous function, by fundamental theorem of calculus, we have $\frac{\mathrm{d}g(s)}{\mathrm{d}s} = c_1 + \phi h(s)$. Note that (4) implies $h(s) \geq g(s)$. Furthermore, since $\phi > 0$,

$$\begin{aligned} \frac{\mathrm{d}(g(s)e^{-\phi s})}{\mathrm{d}s} &= \frac{\mathrm{d}g(s)}{\mathrm{d}s}e^{-\phi s} - \phi e^{-\phi s}g(s) \\ &= c_1 e^{-\phi s} + \phi e^{-\phi s}(h(s) - g(s)) \geq c_1 e^{-\phi s}. \end{aligned} \quad (6)$$

By integrating both left- and far right-hand sides of the inequality (6) over the interval $[0, t]$, we get $g(t)e^{-\phi t} - g(0) \geq \frac{c_1}{\phi}(1 - e^{-\phi t})$. By using this inequality and $g(0) = c_0$, we obtain $g(t) \geq c_0 e^{\phi t} + \frac{c_1}{\phi}(e^{\phi t} - 1)$ for $t \geq 0$, which implies

(5), since $h(t) \geq g(t)$. Finally, if (4) holds with equality, then $h(s) = g(s)$ for $s \geq 0$, and thus, (6) holds with equality, which implies that (5) holds with equality. $\qquad\square$

III. CONDITIONS FOR IMPOSSIBILITY OF STABILIZATION

In this section, we present our main result, which provides conditions on the control constraint (3), under which the stochastic system (1) is impossible to be stabilized.

Theorem 3.1: Consider the stochastic system (1). Assume that there exist a nonnegative-definite Hermitian matrix $R \in \mathbb{C}^{n \times n} \setminus \{0\}$ and a scalar $\phi_{\mathrm{L}} > 0$ such that

$$A^{\mathrm{T}} R + RA + \sum_{i=1}^{\ell_1} C_i^{\mathrm{T}} R C_i \geq \phi_{\mathrm{L}} R, \tag{7}$$

$$\mathrm{tr}\left(D^{\mathrm{T}} R D\right) > 0. \tag{8}$$

If the control policy is $\mathcal{F}_t$-adapted and satisfies (3) with

$$\hat{u} < \begin{cases} \phi_{\mathrm{L}} \mathrm{tr}\left(D^{\mathrm{T}} R D\right)/\beta_{\mathrm{U}}, & \text{if } \beta_{\mathrm{U}} \neq 0, \\ \infty, & \text{otherwise,} \end{cases} \tag{9}$$

where $\beta_{\mathrm{U}} \triangleq \lambda_{\max}(B^{\mathrm{T}} R B)$, then the second moment of the state diverges, that is,

$$\lim_{t \to \infty} \mathbb{E}[\|x(t)\|^2] = \infty, \tag{10}$$

for any initial state $x_0 \in \mathbb{R}^n$.

Proof: Let $V(x) \triangleq x^{\mathrm{T}} R x$. As a first step, we will show $\lim_{t \to \infty} \mathbb{E}[V(x(t))] = \infty$. Let $\Lambda \triangleq A^{\mathrm{T}} R + RA + \sum_{i=1}^{\ell_1} C_i^{\mathrm{T}} R C_i$. It follows from Ito formula (see Section 4.2 of [20]) that

$$\begin{aligned} \mathrm{d}V(x(t)) &= \mathrm{tr}(D^{\mathrm{T}} R D)\mathrm{d}t + x^{\mathrm{T}}(t)\Lambda x(t)\mathrm{d}t \\ &\quad + (x^{\mathrm{T}}(t)RBu(t) + u^{\mathrm{T}}(t)B^{\mathrm{T}} R x(t))\mathrm{d}t \\ &\quad + \sum_{i=1}^{\ell_1} x^{\mathrm{T}}(t)(C_i^{\mathrm{T}} R + R C_i)x(t)\mathrm{d}W_i(t) \\ &\quad + \sum_{i=1}^{\ell_2} (x^{\mathrm{T}}(t)R d_i + d_i^{\mathrm{T}} R x(t))\mathrm{d}W_{i+\ell_1}(t), \end{aligned} \tag{11}$$

where $d_i \in \mathbb{R}^n$, $i \in \{1, \ldots, \ell_2\}$, denote the columns of matrix D. Under an $\mathcal{F}_t$-adapted control policy, $\{x(t)\}_{t \geq 0}$ is $\mathcal{F}_t$-adapted. Thus, by Theorem 3.2.1 of [20], we have $\mathbb{E}[\int_0^t x^{\mathrm{T}}(\tau)(C_i^{\mathrm{T}} R + R C_i)x(\tau)\mathrm{d}W_i(\tau)] = 0$ and $\mathbb{E}[\int_0^t (x^{\mathrm{T}}(\tau)R d_i + d_i^{\mathrm{T}} R x(\tau))\mathrm{d}W_{i+\ell_1}(\tau)] = 0$. As a result, it follows from (11) that

$$\begin{aligned} &\mathbb{E}[V(x(t))] \\ &= \mathbb{E}[V(x(0))] + \mathrm{tr}(D^{\mathrm{T}} R D)t + \mathbb{E}\left[\int_0^t x^{\mathrm{T}}(\tau)\Lambda x(\tau)\mathrm{d}\tau\right] \\ &\quad + \mathbb{E}\left[\int_0^t (x^{\mathrm{T}}(\tau)RBu(\tau) + u^{\mathrm{T}}(\tau)B^{\mathrm{T}} R x(\tau))\mathrm{d}\tau\right], \end{aligned} \tag{12}$$

for $t \geq 0$. Next, we change the order of expectation and integration in (12) by using Fubini's theorem [21] and obtain

$$\begin{aligned} &\mathbb{E}[V(x(t))] \\ &= \mathbb{E}[V(x(0))] + \mathrm{tr}(D^{\mathrm{T}} R D)t + \int_0^t \mathbb{E}[x^{\mathrm{T}}(\tau)\Lambda x(\tau)]\mathrm{d}\tau \\ &\quad + \int_0^t \mathbb{E}[x^{\mathrm{T}}(\tau)RBu(\tau) + u^{\mathrm{T}}(\tau)B^{\mathrm{T}} R x(\tau)]\mathrm{d}\tau. \end{aligned} \tag{13}$$

In what follows, we use (13) to show $\lim_{t \to \infty} \mathbb{E}[V(x(t))] = \infty$, separately for two cases: $\beta_{\mathrm{U}} = 0$ and $\beta_{\mathrm{U}} > 0$.

First, consider the case where $\beta_{\mathrm{U}} = \lambda_{\max}(B^{\mathrm{T}} R B) = 0$. In this case, we have $R^{\frac{1}{2}} B = 0$, and hence $RB = 0$. Furthermore, (7) implies $\mathbb{E}[x^{\mathrm{T}}(\tau)\Lambda x(\tau)] \geq \phi_{\mathrm{L}} \mathbb{E}[x^{\mathrm{T}}(\tau)R x(\tau)] = \phi_{\mathrm{L}} \mathbb{E}[V(x(\tau))]$. As a consequence, we obtain from (13) that $\mathbb{E}[V(x(t))] \geq \mathbb{E}[V(x(0))] + \mathrm{tr}(D^{\mathrm{T}} R D)t + \phi_{\mathrm{L}} \int_0^t \mathbb{E}[V(x(\tau))]\mathrm{d}\tau$ for all $t \geq 0$. Therefore, we can use Lemma 2.3 with $c_0 = \mathbb{E}[V(x(0))]$, $c_1 = \mathrm{tr}(D^{\mathrm{T}} R D)$, $\phi = \phi_{\mathrm{L}}$, and $h(t) = \mathbb{E}[V(x(t))]$ to obtain

$$\begin{aligned} \mathbb{E}[V(x(t))] &\geq \mathbb{E}[V(x(0))]e^{\phi_{\mathrm{L}} t} \\ &\quad + (\mathrm{tr}(D^{\mathrm{T}} R D)/\phi_{\mathrm{L}})(e^{\phi_{\mathrm{L}} t} - 1). \end{aligned} \tag{14}$$

Notice that ϕ_{L} is positive, and hence, $\lim_{t \to \infty} e^{\phi_{\mathrm{L}} t} = \infty$. Moreover, $\mathrm{tr}(D^{\mathrm{T}} R D)$ is positive by the assumption (8). As a result, (14) implies $\lim_{t \to \infty} \mathbb{E}[V(x(t))] = \infty$.

Next, we will show that $\lim_{t \to \infty} \mathbb{E}[V(x(t))] = \infty$ holds also for the case where $\beta_{\mathrm{U}} > 0$. For this case let $\kappa \triangleq \phi_{\mathrm{L}} \mathrm{tr}\left(D^{\mathrm{T}} R D\right)/\beta_{\mathrm{U}}$ and $\mathcal{Q} \triangleq \{q > 1 : \hat{u} < \kappa/q\}$. By Lemma 2.2, we have $\mathcal{Q} \neq \emptyset$.

Now let $\hat{q} \in \mathcal{Q}$ and define $\gamma(\hat{q}) \triangleq \hat{q}/\phi_{\mathrm{L}}$. The scalars $\gamma^{1/2}(\hat{q})$ and $\gamma^{-1/2}(\hat{q})$ are well-defined since $\gamma(\hat{q}) > 0$. Moreover, since R is a nonnegative-definite Hermitian matrix, we have $0 \leq z^{\mathrm{T}} R z$ for any $z \in \mathbb{R}^n$. Using this inequality with $z = \gamma^{-1/2}(q)x(\tau) + \gamma^{1/2}(q)Bu(\tau)$, we get

$$\begin{aligned} 0 &\leq \left(\gamma^{-1/2}(q)x(\tau) + \gamma^{1/2}(q)Bu(\tau)\right)^{\mathrm{T}} R \\ &\quad \cdot \left(\gamma^{-1/2}(q)x(\tau) + \gamma^{1/2}(q)Bu(\tau)\right) \\ &= \gamma^{-1}(\hat{q})x^{\mathrm{T}}(\tau)R x(\tau) + x^{\mathrm{T}}(\tau)RBu(\tau) \\ &\quad + u^{\mathrm{T}}(\tau)B^{\mathrm{T}} R x(\tau) + \gamma(\hat{q})u_1^{\mathrm{T}}(\tau)B^{\mathrm{T}} RBu(\tau), \end{aligned}$$

which implies

$$\begin{aligned} &x^{\mathrm{T}}(\tau)RBu(\tau) + u^{\mathrm{T}}(\tau)B^{\mathrm{T}} R x(\tau) \\ &\geq -\gamma^{-1}(\hat{q})x^{\mathrm{T}}(\tau)R x(\tau) - \gamma(\hat{q})u^{\mathrm{T}}(\tau)B^{\mathrm{T}} RBu(\tau). \end{aligned} \tag{15}$$

It then follows from (13) together with $\mathbb{E}[x^{\mathrm{T}}(\tau)\Lambda x(\tau)] \geq \phi_{\mathrm{L}} \mathbb{E}[V(x(\tau))]$ and (15) that

$$\begin{aligned} \mathbb{E}[V(x(t))] &\geq \mathbb{E}[V(x(0))] + \mathrm{tr}(D^{\mathrm{T}} R D)t \\ &\quad + (\phi_{\mathrm{L}} - \gamma^{-1}(\hat{q})) \int_0^t \mathbb{E}[V(x(\tau))]\mathrm{d}\tau \\ &\quad - \gamma(\hat{q}) \int_0^t \mathbb{E}[u^{\mathrm{T}}(\tau)B^{\mathrm{T}} RBu(\tau)]\mathrm{d}\tau. \end{aligned} \tag{16}$$

Since $\gamma(\hat{q}) > 0$, we have $-\gamma(\hat{q}) < 0$. Thus, by using $u^{\mathrm{T}}(\tau)B^{\mathrm{T}} RBu(\tau) \leq \lambda_{\max}(B^{\mathrm{T}} R B)\|u(\tau)\|^2 = \beta_{\mathrm{U}}\|u(\tau)\|^2$ with (16), we obtain

$$\begin{aligned} \mathbb{E}[V(x(t))] &\geq \mathbb{E}[V(x(0))] + \mathrm{tr}(D^{\mathrm{T}} R D)t \\ &\quad + (\phi_{\mathrm{L}} - \gamma^{-1}(\hat{q})) \int_0^t \mathbb{E}[V(x(\tau))]\mathrm{d}\tau \\ &\quad - \gamma(\hat{q})\beta_{\mathrm{U}} \int_0^t \mathbb{E}[\|u(\tau)\|^2]\mathrm{d}\tau. \end{aligned} \tag{17}$$

Now, since the control policy satisfies (3), we have $\int_0^t \mathbb{E}[\|u(\tau)\|^2]\mathrm{d}\tau \le \hat{u}t$, and hence, it follows from (17) that

$$
\begin{aligned}
\mathbb{E}[V(x(t))] \ge{} & \mathbb{E}[V(x(0))] + \mathrm{tr}(D^{\mathrm{T}}RD)t \\
& + (\phi_{\mathrm{L}} - \gamma^{-1}(\hat{q}))\int_0^t \mathbb{E}[V(x(\tau))]\mathrm{d}\tau \\
& - \gamma(\hat{q})\beta_{\mathrm{U}}\hat{u}t, \quad t \ge 0.
\end{aligned} \tag{18}
$$

Let $c_0 \triangleq \mathbb{E}[V(x(0))]$, $c_1 \triangleq \mathrm{tr}(D^{\mathrm{T}}RD) - \gamma(\hat{q})\beta_{\mathrm{U}}\hat{u}$, $\phi \triangleq \phi_{\mathrm{L}} - \gamma^{-1}(\hat{q})$, and $h(t) \triangleq \mathbb{E}[V(x(t))]$. By definition of $\mathcal{Q}$, we have $\hat{q} > 1$, which implies $1 - 1/\hat{q} > 0$. This inequality and $\phi_{\mathrm{L}} > 0$ imply $\phi = \phi_{\mathrm{L}} - \gamma^{-1}(\hat{q}) = \phi_{\mathrm{L}}(1 - 1/\hat{q}) > 0$. Thus, by Lemma 2.3, we obtain

$$
\mathbb{E}[V(x(t))] \ge c_0 e^{\phi t} + (c_1/\phi)(e^{\phi t} - 1). \tag{19}
$$

Since V is a nonnegative-definite function, we have $c_0 \ge 0$. Next, we show $c_1 > 0$. By definition of $\mathcal{Q}$, we have $\hat{u} < \phi_{\mathrm{L}}\mathrm{tr}(D^{\mathrm{T}}RD)/(\beta_{\mathrm{U}}\hat{q})$. Noting that $\gamma(\hat{q}) = \hat{q}/\phi_{\mathrm{L}}$, this inequality implies $\gamma(\hat{q})\beta_{\mathrm{U}}\hat{u} < \mathrm{tr}(D^{\mathrm{T}}RD)$. Thus, $c_1 = \mathrm{tr}(D^{\mathrm{T}}RD) - \gamma(\hat{q})\beta_{\mathrm{U}}\hat{u} > 0$. Now, since $c_0 \ge 0$, $c_1 > 0$, and $\phi > 0$ hold, (19) implies $\lim_{t\to\infty}\mathbb{E}[V(x(t))] = \infty$.

Finally, since $R \in \mathbb{C}^{n\times n} \setminus \{0\}$ (i.e., $R \ne 0$), the nonnegative-definite Hermitian matrix R has at least one eigenvalue strictly larger than 0. Thus, $\lambda_{\max}(R) > 0$. Consequently, $V(x(t)) \le \lambda_{\max}(R)\|x(t)\|^2$ implies $\|x(t)\|^2 \ge (1/\lambda_{\max}(R))V(x(t))$ for $t \ge 0$. Hence, (10) follows from $\lim_{t\to\infty}\mathbb{E}[V(x(t))] = \infty$. $\square$

Theorem 3.1 provides sufficient conditions under which the system (1) is instabilizable and the second moment of the state diverges regardless of the controller design and the initial state value. Condition (7) in Theorem 3.1 quantifies the instability of the uncontrolled ($u(t) \equiv 0$) system, and the term $\mathrm{tr}(D^{\mathrm{T}}RD)$ in (8) represents the effect of additive noise characterized with the matrix D. If there is no multiplicative noise (i.e., $C_i = 0$ for $i \in \{1, \ldots, \ell_1\}$), then (7) requires A to be strictly unstable. On the other hand, when there is multiplicative noise, (7) may hold even if A is Hurwitz-stable. Notice also that R is a nonnegative-definite matrix and it may have 0 as an eigenvalue. This property is essential in our analysis, since it allows us to deal with the cases where some of the states are diverging, while the others are stable.

Theorem 3.1 implies that if conditions (7), (8) hold, then it is *not possible* to stabilize the system by using control inputs with too small average second moments as in (3). The impossibility threshold on the average second moment of control input $u(t)$ is characterized in (9). If $\lambda_{\max}(B^{\mathrm{T}}RB) = 0$, then this threshold value becomes infinity indicating that stabilization is impossible regardless of the input constraint. We note that the case $\lambda_{\max}(B^{\mathrm{T}}RB) = 0$ represents the situation, where $u(t)$ does not have any effect on $x^{\mathrm{T}}(t)Rx(t)$.

Remark 3.2 (Instability conditions): The structure of condition (7) is similar to those of stability/instability conditions provided in [22] for stochastic systems with multiplicative noise. In particular, when specialized to linear systems, Corollary 4.7 of [22] yields an instability condition based on existence of a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ and a scalar $\psi > 0$ such that $A^{\mathrm{T}}P + PA + \sum_{i=1}^{\ell_1}C_i^{\mathrm{T}}PC_i \ge \psi P$. Notice

that for systems with only multiplicative noise, a positive-definite matrix P is required to show global instability. In our setting, a nonnegative-definite matrix R is sufficient, because there is also additive noise and (8) guarantees that this noise can make the projection of the state on unstable modes of the uncontrolled system take a nonzero value even if the initial state is zero. Moreover, under the condition (9), $\mathbb{E}[x^{\mathrm{T}}(t)Rx(t)]$ diverges, which in turn implies divergence of the second moment of the state, as shown in the proof of Theorem 3.1. ◁

Remark 3.3 (Numerical approach): We note that linear matrix inequalities can be used for checking the conditions of Theorem 3.1. First of all, for a given ϕ_{L}, condition (7) is linear in R. Similarly, (8) is a linear inequality of R. Note that (8) also guarantees that $R \ne 0$. Moreover, the inequality (9) can be transformed into $\overline{\beta}\hat{u} < \phi_{\mathrm{L}}\mathrm{tr}(D^{\mathrm{T}}RD)$ and $B^{\mathrm{T}}RB \le \overline{\beta}I_m$, which are linear in R and $\overline{\beta} \ge 0$, for a given ϕ_{L}. If the abovementioned inequalities are satisfied with $R = \widetilde{R}$, then $R = c\widetilde{R}$ with any $c > 0$ also satisfies them. To restrict the solutions, we can impose an additional constraint $\mathrm{tr}(R) = 1$. In our numerical method, we iterate over a set of candidate values of ϕ_{L} and utilize linear matrix inequality solvers (for each value of ϕ_{L}) to check the conditions of Theorem 3.1. ◁

Remark 3.4 (Partial constraints): Theorem 3.1 can be extended to handle partial input constraints. Consider $\mathrm{d}x(t) = (Ax(t) + Bu(t) + F\nu(t))\mathrm{d}t + [\Psi(x(t)), D]\,\mathrm{d}W(t)$, where $u(t)$ is constrained as in (3) and $\nu(t)$ is *unconstrained*. If (7)–(9) and $RF = 0$ hold, then it is impossible to achieve stabilization of this modified system. The proof is similar to that of Theorem 3.1, as $RF = 0$ implies that $V(x(t)) = x^{\mathrm{T}}(t)Rx(t)$ is not affected by $\nu(\cdot)$, and hence (11) holds. ◁

A. Tightness of the result for scalar systems

Theorem 3.1 provides a tight bound for $\hat{u}$ in (9) for scalar systems with a scalar state and a scalar constrained input.

Consider (1) with scalars A, B, D and scalar-valued function $\Psi(x) \triangleq C_1 x$ such that $2A + C_1^2 > 0$, $B \ne 0$, and $D \ne 0$. In this case, conditions (7) and (8) hold with $R = 1$ and $\phi_{\mathrm{L}} = 2A + C_1^2$. Thus, Theorem 3.1 implies that if the control policy satisfies (3) with $\hat{u} < (2A + C_1^2)D^2/B^2$, then the system is impossible to stabilize regardless of the initial state x_0. This bound is tight, because, as shown in the following result, stability can be achieved when $\hat{u} = (2A + C_1^2)D^2/B^2$.

Proposition 3.5: Consider (1) with scalars $A, B, D \in \mathbb{R}$ and scalar-valued function $\Psi(x) \triangleq C_1 x$. Suppose $2A + C_1^2 > 0$, $B \ne 0$, $D \ne 0$, and $x_0 = 0$. Then feedback control policy $u(t) = Kx(t)$ with $K = -(2A + C_1^2)/B$ can achieve stabilization (i.e., $\sup_{t\ge 0}\mathbb{E}[x^2(t)] < \infty$) and satisfies (3) with $\hat{u} = (2A + C_1^2)D^2/B^2$.

Proof: Let $\Lambda \triangleq 2(A + BK) + C_1^2$ and $\phi(x, t) \triangleq e^{-\Lambda t}x^2$. By Ito formula (Section 4.2 of [20]),

$$
\begin{aligned}
\mathrm{d}\phi(x(t), t) ={} & 2e^{-\Lambda t}C_1 x^2(t)\mathrm{d}W_1(t) + 2e^{-\Lambda t}Dx(t)\mathrm{d}W_2(t) \\
& + e^{-\Lambda t}D^2\mathrm{d}t.
\end{aligned} \tag{20}
$$

Now, since $\{x(t)\}_{t\ge 0}$ is an $\mathcal{F}_t$-adapted process, we obtain $\mathbb{E}[\int_0^t 2e^{-\Lambda\tau}C_1 x^2(\tau)\mathrm{d}W_1(\tau)] = 0$ and $\mathbb{E}[\int_0^t 2e^{-\Lambda\tau}Dx(\tau)\mathrm{d}W_2(\tau)] = 0$, by using Theorem 3.2.1 of [20]. Thus, with $x_0 = 0$, (20) implies $\mathbb{E}[\phi(x(t), t)] =$

$\mathbb{E}[\phi(x(0),0)] + \int_0^t e^{-\Lambda\tau}D^2\mathrm{d}\tau = -(D^2/\Lambda)(e^{-\Lambda t} - 1)$. Therefore, for all $t \geq 0$, we have $\mathbb{E}[x^2(t)] = e^{\Lambda t}\mathbb{E}[\phi(x(t),t)] = -(D^2/\Lambda)(1 - e^{\Lambda t}) \leq -D^2/\Lambda$, where the last inequality follows from $\Lambda = -(2A + C_1^2) < 0$. As a consequence, $\sup_{t\geq 0}\mathbb{E}[x^2(t)] \leq -D^2/\Lambda < \infty$, which implies that stability is achieved. Moreover, we have $\mathbb{E}[u^2(t)] = K^2\mathbb{E}[x^2(t)] \leq (2A + C_1^2)D^2/B^2$ for all $t \geq 0$, showing that control input constraint (3) is satisfied with $\hat{u} = (2A + C_1^2)D^2/B^2$. $\square$

Proposition 3.5 handles the case where $2A + C_1^2 > 0$. With similar analysis, we can also show that if $2A + C_1^2 < 0$, then $\mathbb{E}[x^2(t)]$ stays bounded even without control (i.e., $u(t) \equiv 0$). Furthermore, if $2A + C_1^2 = 0$, $B \neq 0$, then a state-feedback control policy $u(t) = Kx(t)$ with $BK < 0$ achieves stabilization, and moreover, for $x_0 = 0$, it guarantees the bound $\mathbb{E}[u^2(t)] \leq -(1/2)D^2K/B$ for all $t \geq 0$. This bound can be made arbitrarily small by choosing small $|K|$. If $2A + C_1^2 = 0$, $B = 0$, then the system is uncontrollable and $\mathbb{E}[x^2(t)]$ grows unboundedly unless $D = 0$.

B. Existence of instability-inducing noise matrices

The following proposition complements Theorem 3.1. It shows that if $A, C_1, \ldots, C_{\ell_1}$ satisfy (7), then for any B and $\hat{u}$, there exists a noise matrix D that satisfies both (8) and (9). Thus, by Theorem 3.1, the system with that noise matrix is impossible to be stabilized under constraint (3).

Proposition 3.6: Assume that there exist a nonnegative-definite Hermitian matrix $R \in \mathbb{C}^{n\times n}\setminus\{0\}$ and a scalar $\phi_{\mathrm{L}} > 0$ that satisfy (7). Then for any $B \in \mathbb{R}^{n\times m}$ and $\hat{u} \in [0,\infty)$, there exists $D \in \mathbb{R}^{n\times\ell_2}$ such that both (8) and (9) hold.

Proof: By the spectral theorem for Hermitian matrices (see Theorem 2.5.6 of [23]), the nonnegative-definite Hermitian matrix R can be written as $R = \Xi\,\mathrm{diag}(\mu_1,\ldots,\mu_n)\Xi^*$, where $\mu_1,\ldots,\mu_n \geq 0$ are the eigenvalues of R and $\Xi \in \mathbb{C}^{n\times n}\setminus\{0\}$ is a unitary matrix. Let $\xi_1,\ldots,\xi_n \in \mathbb{C}^n\setminus\{0\}$ denote the columns of Ξ. We have $R = \sum_i \mu_i\xi_i\xi_i^*$. Since $R \neq 0$, at least one eigenvalue of R is strictly larger than 0. Let $\tilde{i} \triangleq \min\{i \in \{1,\ldots,n\}: \mu_i > 0\}$. Now, let $\xi_{\tilde{i},j} \in \mathbb{C}$ denote the jth entry of vector $\xi_{\tilde{i}}$. Since $\xi_{\tilde{i}} \neq 0$, at least one entry of $\xi_{\tilde{i}}$ is nonzero. We define $\tilde{j} \triangleq \min\{j \in \{1,\ldots,n\}: \xi_{\tilde{i},j} \neq 0\}$. Now let $\alpha > 0$ and $D \triangleq [d_1,\ldots,d_{\ell_2}]$, where $d_i \in \mathbb{R}^n$, $i \in \{1,\ldots,\ell_2\}$, are columns of D. We let $d_i = 0$ for $i \neq 1$, and set the entries of $d_1 \in \mathbb{R}^n$ as

$$d_{1,j} = \begin{cases} \dfrac{(\alpha+(\hat{u}\beta_{\mathrm{U}}/\phi_{\mathrm{L}}))^{1/2}}{\mu_{\tilde{i}}^{1/2}|\xi_{\tilde{i},\tilde{j}}|}, & \text{if } j = \tilde{j}, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where $\beta_{\mathrm{U}} = \lambda_{\max}(B^{\mathrm{T}}RB)$. Since $d_i = 0$ for $i \neq 1$, we have

$$\mathrm{tr}(D^{\mathrm{T}}RD) = \mathrm{tr}\left(\sum_{i=1}^n \mu_i D^{\mathrm{T}}\xi_i\xi_i^* D\right)$$
$$= \sum_{i=1}^n \mu_i\mathrm{tr}(D^{\mathrm{T}}\xi_i\xi_i^* D) = \sum_{i=1}^n \mu_i \sum_{j=1}^{\ell_2} \left(d_j^{\mathrm{T}}\xi_i\xi_i^* d_j\right)$$
$$= \sum_{i=1}^n \mu_i d_1^{\mathrm{T}}\xi_i\xi_i^* d_1 = d_1^{\mathrm{T}}\left(\sum_{i=1}^n \mu_i\xi_i\xi_i^*\right)d_1. \tag{22}$$

Moreover, since $\sum_{i=1}^n \mu_i\xi_i\xi_i^* \geq \mu_{\tilde{i}}\xi_{\tilde{i}}\xi_{\tilde{i}}^*$, it follows from (22) that $\mathrm{tr}(D^{\mathrm{T}}RD) \geq \mu_{\tilde{i}}d_1^{\mathrm{T}}\xi_{\tilde{i}}\xi_{\tilde{i}}^* d_1$. By using this inequality and noting that $\mu_{\tilde{i}} > 0$, we obtain from (21) that

$$\mathrm{tr}(D^{\mathrm{T}}RD) \geq \mu_{\tilde{i}}\left(\sum_{j=1}^n d_{1,j}\xi_{\tilde{i},j}\right)\left(\sum_{j=1}^n \overline{\xi}_{\tilde{i},j}d_{1,j}\right)$$
$$= \mu_{\tilde{i}}\xi_{\tilde{i},\tilde{j}}\overline{\xi}_{\tilde{i},\tilde{j}}d_{1,\tilde{j}}^2 = \mu_{\tilde{i}}|\xi_{\tilde{i},\tilde{j}}|^2\frac{\alpha + (\hat{u}\beta_{\mathrm{U}}/\phi_{\mathrm{L}})}{\mu_{\tilde{i}}|\xi_{\tilde{i},\tilde{j}}|^2}$$
$$= \alpha + (\hat{u}\beta_{\mathrm{U}}/\phi_{\mathrm{L}}), \tag{23}$$

which implies (8), as $\alpha > 0$, $\hat{u} \geq 0$, $\beta_{\mathrm{U}} \geq 0$, and $\phi_{\mathrm{L}} > 0$. If $\beta_{\mathrm{U}} = 0$, then the inequality (9) holds directly, as $\hat{u} < \infty$. If $\beta_{\mathrm{U}} \neq 0$, then the inequality (23) implies $\mathrm{tr}(D^{\mathrm{T}}RD) > \hat{u}\beta_{\mathrm{U}}/\phi_{\mathrm{L}}$, which in turn implies (9). $\square$

IV. SPECIAL SETTING WITH ONLY ADDITIVE NOISE

In this section, we are interested in a special setting, where $\Psi(x(t)) = 0$ in (1). In this setting, the system does not face multiplicative noise and it is only subject to additive noise.

We first present an improvement of the numerical approach presented in Remark 3.3. Then we show that eigenstructure of A can be used to obtain new instability conditions that are easier to check compared to Theorem 3.1. The eigenstructure-based analysis was previously considered only for discrete-time systems in [16]. Here, we show that continuous-time systems also allow a similar approach.

A. Candidate values of ϕ_{L} in instability analysis

Remark 3.3 provides a numerical approach for checking instability conditions of Theorem 3.1. This approach is based on iterating over a set of candidate values of ϕ_{L} and checking the feasibility of certain linear matrix inequalities. In the general case with both additive and multiplicative noise, we do not have prespecified bounds for the candidate value set. However, in the case of only additive noise ($\Psi(x(t)) = 0$, i.e., $C_i = 0$ in (2)), the candidate values of ϕ_{L} can be restricted to belong to the set $(0, 2\vartheta_{\max}(A)]$, where we define $\vartheta_{\max}: \mathbb{C}^{n\times n} \to \mathbb{R}$ as

$$\vartheta_{\max}(A) \triangleq \max\{\mathrm{Re}(\lambda): \lambda \in \mathrm{spec}(A)\} \tag{24}$$

with $\mathrm{spec}(A) \subset \mathbb{C}$ denoting the set of eigenvalues of A. It is sufficient to choose the values of ϕ_{L} from the set $(0, 2\vartheta_{\max}(A)]$, as it is not possible to satisfy (7) with $\phi_{\mathrm{L}} > 2\vartheta_{\max}(A)$ and $R \neq 0$, as shown in the following proposition. Here, we note that with $C_i = 0$, $i \in \{1,\ldots,\ell_1\}$, the inequality (7) reduces to $A^{\mathrm{T}}R + RA \geq \phi_{\mathrm{L}}R$.

Proposition 4.1: Let $A \in \mathbb{C}^{n\times n}$. For every nonnegative-definite Hermitian matrix $R \in \mathbb{C}^{n\times n}\setminus\{0\}$, there exists $y \in \mathbb{C}^n\setminus\{0\}$ such that $R^{\frac{1}{2}}y \neq 0$ and $y^*(A^*R + RA)y \leq 2\vartheta_{\max}(A)y^*Ry$, where $\vartheta_{\max}(A)$ is defined in (24).

Proof: It follows from Lemma A.1 of [16] that $R^{\frac{1}{2}}A\hat{\nu} = \hat{\lambda}R^{\frac{1}{2}}\hat{\nu}$, where $\hat{\lambda} \in \mathbb{C}$ is an eigenvalue of A and $\hat{\nu} \in \mathbb{C}^n\setminus\{0\}$ is a generalized eigenvector of A that satisfies $R^{\frac{1}{2}}\hat{\nu} \neq 0$. Let $y \triangleq \hat{\nu}$. We have $R^{\frac{1}{2}}y \neq 0$. Moreover, $y^*(A^*R + RA)y = \overline{\hat{\lambda}}y^*Ry + \hat{\lambda}y^*Ry = 2\mathrm{Re}(\hat{\lambda})y^*Ry$. Therefore, $y^*(A^*R + RA)y \leq 2\vartheta_{\max}(A)y^*Ry$, since $\mathrm{Re}(\hat{\lambda}) \leq \vartheta_{\max}(A)$. $\square$

B. Instability conditions based on the eigenstructure of A

Even with the improvement discussed in the previous subsection, checking feasibility of the linear matrix inequalities mentioned in Remark 3.3 can be computationally costly. In this subsection, we show that the eigenstructure of the system matrix A can be used to derive instability conditions that can be checked numerically efficiently.

Let $r \in \{1, \ldots, n\}$ denote the number of distinct eigenvalues of A and let $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{C}$ with $\lambda_i \neq \lambda_j$ denote those eigenvalues. Moreover, for every $i \in \{1, \ldots, r\}$, let $n_i \in \{1, \ldots, n\}$ represent the geometric multiplicity of the eigenvalue λ_i. Eigenvalues of A are also the eigenvalues of A^{T} with the same multiplicities. Thus, for every λ_i, there exists n_i number of vectors $v_{i,j} \in \mathbb{C}^n$ such that

$$A^{\mathrm{T}} v_{i,j} = \lambda_i v_{i,j}, \quad j \in \{1, \ldots, n_i\}. \qquad (25)$$

These vectors $v_{i,1}, \ldots, v_{i,n_i}$ are called the left-eigenvectors of A associated with the eigenvalue λ_i. We remark that if λ_i is a complex eigenvalue (i.e., $\lambda_i \notin \mathbb{R}$), then the complex conjugate $\overline{\lambda_i}$ is also an eigenvalue of A, and moreover,

$$v_{i,j}^* A = \overline{\lambda_i} v_{i,j}^*, \quad j \in \{1, \ldots, n_i\}, \qquad (26)$$

where $v_{i,j}^*$ is the complex conjugate transpose of $v_{i,j} \in \mathbb{C}^n$. In the instability conditions presented below, we use the eigenvalues λ_i, and the left-eigenvectors $v_{i,j} \in \mathbb{C}^n$, $j \in \{1, \ldots, n_i\}$, $i \in \{1, \ldots, r\}$. Furthermore, we define

$$\mathcal{I} \triangleq \{(i,j) \colon \mathrm{Re}(\lambda_i) > 0, \ v_{i,j}^* DD^{\mathrm{T}} v_{i,j} > 0,$$
$$j \in \{1, \ldots, n_i\}, i \in \{1, \ldots, r\}\}.$$

Corollary 4.2: Consider the linear stochastic system (1) where $\Psi(x) = 0$. Suppose $\mathcal{I} \neq \emptyset$. If the control policy is $\mathcal{F}_t$-adapted and satisfies (3) with

$$\hat{u} < \max_{(i,j) \in \mathcal{I}} \varphi_{i,j}, \qquad (27)$$

where

$$\varphi_{i,j} \triangleq \begin{cases} 2\mathrm{Re}(\lambda_i) v_{i,j}^* DD^{\mathrm{T}} v_{i,j}/\beta_{\mathrm{U}}^{i,j}, & \text{if } \beta_{\mathrm{U}}^{i,j} \neq 0, \\ \infty, & \text{otherwise}, \end{cases}$$

and $\beta_{\mathrm{U}}^{i,j} \triangleq \lambda_{\max}(B^{\mathrm{T}} v_{i,j} v_{i,j}^* B)$, $(i,j) \in \mathcal{I}$, then the second moment of the state diverges (i.e., (10) holds) for any initial state $x_0 \in \mathbb{R}^n$.

Proof: By (25) and (26), for each $(i,j) \in \mathcal{I}$, we obtain $A^{\mathrm{T}} v_{i,j} v_{i,j}^* + v_{i,j} v_{i,j}^* A = \lambda_i v_{i,j} v_{i,j}^* + \overline{\lambda_i} v_{i,j} v_{i,j}^* = 2\mathrm{Re}(\lambda_i) v_{i,j} v_{i,j}^*$. Since $v_{i,j} v_{i,j}^* \in \mathbb{C}^{n \times n}$ is a nonnegative-definite Hermitian matrix, we have (7) with $R = v_{i,j} v_{i,j}^*$ and $\phi_{\mathrm{L}} = \mathrm{Re}(\lambda_i)$. Moreover, by the definition of $\mathcal{I}$, we have $\mathrm{tr}(D^{\mathrm{T}} v_{i,j} v_{i,j}^* D) = v_{i,j}^* DD^{\mathrm{T}} v_{i,j} > 0$ for $(i,j) \in \mathcal{I}$. Thus, (8) holds with $R = v_{i,j} v_{i,j}^*$. Now, since (7) and (8) both hold for each $(i,j) \in \mathcal{I}$, it follows from Theorem 3.1 by setting $\beta_{\mathrm{U}} = \beta_{\mathrm{U}}^{i,j}$ that under control policies satisfying (3) with $\hat{u} < \varphi_{i,j}$, the second moment of the state diverges. Finally, (27) implies that there exists $(\tilde{i}, \tilde{j}) \in \mathcal{I}$ such that $\hat{u} < \varphi_{\tilde{i}, \tilde{j}}$, implying divergence. $\square$

V. NUMERICAL EXAMPLES

In this section, we illustrate our results on two example continuous-time stochastic systems.

Setting #	a	c_1	c_2	d	Applied Result	Range of $\hat{u}$
1	-0.5	2	2	1	Theorem 3.1	$[0, 7.4)$
2	1.5	2	2	1	Theorem 3.1	$[0, 10.8)$
3	1.5	2	0	1	Theorem 3.1	$[0, 8.4)$
4	1.5	0	2	1	Theorem 3.1	$[0, 4.5)$
5	1.5	0	0	1	Theorem 3.1	$[0, 1)$
					Corollary 4.2	$[0, 0.75)$
6	1.5	0	0	2	Theorem 3.1	$[0, 4.1)$
					Corollary 4.2	$[0, 3)$

TABLE I

RANGES OF $\hat{u}$ FOR WHICH STABILIZATION IS IMPOSSIBLE.

Example 1 : Consider the continuous-time stochastic system described by (1) and (2) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ D = d \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad (28)$$

$$\ell_1 = 2, \ C_1 = \begin{bmatrix} 0 & 0 \\ c_1 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix}, \qquad (29)$$

where $a, c_1, c_2, d \in \mathbb{R}$ are scalar coefficients. The case with $a \geq 0$ corresponds to the linearized dynamics of the forced Van der Pol oscillator (see Section 5.3.1 of [24]).

In each row of Table I, we consider a different setting for $a, c_1, c_2, d \in \mathbb{R}$. Our goal is to obtain ranges of $\hat{u}$ such that the system cannot be stabilized with constrained control policies satisfying (3) with $\hat{u}$ chosen in the given range. To obtain the range for each parameter setting, we apply Theorem 3.1. In addition, when there is no multiplicative noise (i.e., $c_1 = c_2 = 0$), we also apply Corollary 4.2.

Setting 1 in Table I represents the case where A is a Hurwitz-stable matrix. Notice that when A is Hurwitz-stable, without multiplicative noise, the second moment of the state of the uncontrolled system would remain bounded. However, as Table I indicates, under multiplicative noise (with parameters $c_1 = c_2 = 2$), the system is unstable under any constrained control that satisfies (3) with $\hat{u} \in [0, 7.4)$.

Settings 2–6 in Table I represent different scenarios where A is strictly unstable. In each of those settings, different noise parameters are considered. The main observation is that systems with increased noise levels are harder to stabilize with constrained controllers.

Settings 5 and 6 in Table I correspond to the cases where there is no multiplicative noise. In those cases Corollary 4.2 can be applied. Notice that Corollary 4.2 provides smaller ranges for $\hat{u}$ compared to Theorem 3.1. This shows that Corollary 4.2 is conservative. We note that the main advantage of Corollary 4.2 is that its conditions can be checked faster than those of Theorem 3.1.

For checking conditions of Corollary 4.2, we can speedily compute $\max_{(i,j) \in \mathcal{I}} \varphi_{i,j}$ in (27). In particular, the computation yields the analytical expression

$$\max_{(i,j) \in \mathcal{I}} \varphi_{i,j} = \begin{cases} d^2(2-a)a, & a \in [0, 2), \\ 4d^2(\frac{1}{2}a - 1), & a \geq 2, \end{cases} \qquad (30)$$

which is also shown in Fig. 1. Given a and d, if $\hat{u} < \max_{(i,j) \in \mathcal{I}} \varphi_{i,j}$ (i.e., the value is below the surface in Fig. 1), then Corollary 4.2 implies that stabilization of (1) is impossible with a control policy that satisfies (3) with that particular
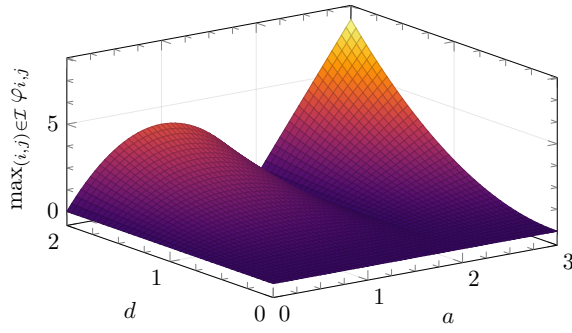
Fig. 1. The value of $\max_{(i,j)\in\mathcal{I}}\varphi_{i,j}$ in (27). If (3) holds with $\hat{u} < \max_{(i,j)\in\mathcal{I}}\varphi_{i,j}$, then stabilization is impossible.

$\hat{u}$. Notice that $\max_{(i,j)\in\mathcal{I}}\varphi_{i,j}$ is a quadratic function of d, and thus, for larger values of d, the value of $\max_{(i,j)\in\mathcal{I}}\varphi_{i,j}$ becomes larger. This result is intuitive in the sense that stabilization becomes harder under stronger noise. On the other hand, $\max_{(i,j)\in\mathcal{I}}\varphi_{i,j}$ depends on a in a nonlinear nonmonotonic way. It follows from (30) that for $a \in [0,2)$, the maximum of $\max_{(i,j)\in\mathcal{I}}\varphi_{i,j}$ is achieved when $a = 1$. For $a \geq 2$, $\max_{(i,j)\in\mathcal{I}}\varphi_{i,j}$ increases as a increases.

Example 2 : Consider (1) and (2) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\zeta^2 & 0 & 0 & 2\zeta \\ 0 & 0 & 0 & 1 \\ 0 & -2\zeta & 0 & 0 \end{bmatrix}, \ B = D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$\ell_1 = 1, \ C_1 = I_4$.

This system is a noisy version of the uncoupled, linearized, and normalized dynamics that describes a satellite's motion in the equatorial plane, as provided in [25]. The scalar ζ is the angular velocity of the equatorial orbit along which the system is linearized and the control input $u(t) \in \mathbb{R}^2$ is the vector of thrusts applied to the satellite in the equatorial plane. We consider the control input constraint (3) with the average second moment bound $\hat{u}$.

We check feasibility of the linear matrix inequalities discussed in Remark 3.3 to assess the conditions of Theorem 3.1 for different values of ζ and $\hat{u}$. When $\zeta = 0.1$, the conditions of Theorem 3.1 hold for $\hat{u} \in [0, 1.7]$. Thus the system is instabilizable under the control constraint (3) with those values of $\hat{u}$. On the other hand, with $\zeta = 1$, the corresponding instabilizability range is obtained as $\hat{u} \in [0, 1.1]$.

VI. CONCLUSION

We have investigated the constrained control problem for linear stochastic systems with additive and multiplicative noise terms. We have shown that in certain scenarios, stabilization is impossible to achieve with control policies that have bounded time-averaged second moments. In particular, we have obtained conditions, under which the second moment of the system state diverges regardless of the controller design and regardless of the initial state. Moreover, we have showed the tightness of our results for scalar systems and provided extensions for partially-constrained control policies and additive-only noise settings.

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