Expander Decomposition and Hierarchies: Exercise 1

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1 Expander + Matching = Expander

Let G be a (connected) ϕ-expander. Let M be a matching where each edge $(u, v) \in M$ has at least one endpoint in $V(G)$. Prove that $G' = G \cup M$ is a $(\phi/2)$-expander.

1.1 Solution

- By definition, $2\deg_G$ is $(\phi/2)$-expanding in G.
- Since $\deg_{G'} \leq \deg_G + 1_V \leq 2\deg_G$, $\deg_{G'}$ is $(\phi/2)$-expanding in G.
- $\deg_{G'}$ is also $(\phi/2)$-expanding in G' because $G \subseteq G'$. So, G' is a $(\phi/2)$-expander.

2 Dynamic expander decomposition

We will use this in Lecture 3. In Lecture 1, we show the existence of expander decomposition. Let us extend this to the dynamic setting.

The Setting. Let $G = (V, E)$ be a fixed graph and ϕ be a parameter. Let $D \subseteq E$ be a set of edges. Let A be a node weighting. Both D and A undergo *incremental* updates. That is, at each time step, the adversary can add new edges into the set D. Also, the adversary can increase the value $A(v)$ for any vertex v.

Exercise. Show the existence of the procedure $\mathrm{DynED}(G, \phi, A, D)$ that maintains an *incremental* edge set $C \supseteq D$ such that, at any time step t,

- $A^{(t)}$ is ϕ-expanding in $G - C^{(t)}$, and
- $|C^{(t)}| - |D^{(t)}| \leq \phi|A^{(t)}| \log(n)$.

Hint: you do not need to use the single payment scheme that works for all time steps t.

3 Boundary-linked expander decomposition

We will use this in Lecture 3. Let $G = (V, E)$ be a graph, A be a node weighting, and a parameter $\beta \leq \frac{1}{4\phi \log(n)}$. Prove that there exists C where

- $A + \beta \deg_C$ is ϕ-expanding in $G - C$, and
- $|C| \leq 2\phi |A| \log(n)$.

Hint: This generalizes the same statement in Lecture 1 when $\beta = 1$. The same proof works.

4 Vertex sparsifier cuts

Flow Sparsifiers for Node Weighting. Recall this definition from Lecture 1. Given a graph $G = (V, E)$ and a node weighting A, we say that H is a *flow vertex-sparsifier for A in G with quality q* if, for

- If D is routable in G, then D is routable in H.
- If D is routable in H, then D is routable in G with congestion q.

Flow Sparsifiers for Terminal Sets. For any terminal set $U \subseteq V$, let A_U be a node weighting where, for every vertex v,

$$A_U(v) = \begin{cases} \deg_G(v) & \text{if } v \in U, \\ 0 & \text{if } v \notin U. \end{cases}$$

We say that H is a *flow vertex-sparsifier for U* in G if H is a flow vertex-sparsifier for A_U.

Cut Sparsifiers for Terminal Sets. We say that H is a *cut vertex-sparsifier for U in G* with quality q if, for every pair of disjoint subsets $X, Y \subseteq U$,

$$\text{mincut}_G(X, Y) \leq \text{mincut}_H(X, Y) \leq q \cdot \text{mincut}_G(X, Y).$$

That is, H q-approximately preserves all minimum cuts between *every subset* of terminals.

Exercise. Prove the following.

1. If H is a flow vertex-sparsifier for U in G with quality q, then H is a cut vertex-sparsifier for U in G with quality q.
2. Using material from Lecture 1, conclude that there exists a cut vertex-sparsifier H for U in G with quality $4 \log n$ and of size $|E(H)| = O(\deg_G(U) \log^2 n)$.

4.1 Solution

1. Let H be a flow vertex-sparsifier for U in G with quality q. Let $X, Y \subseteq U$ be disjoint subsets of terminals.

   - Consider the (X,Y)-maxflow F_G in G. The path decomposition of F_G routes an A_U-respecting demand D_G. Since F_G routes D_G in G with congestion 1, D_G is routable by some flow F_H in H with congestion 1 too. Since F_H routes $\mathrm{mincut}_G(X,Y)$ units of flow from X to Y in H. So $\mathrm{mincut}_H(X,Y) \geq \mathrm{mincut}_G(X,Y)$.
   - Consider the (X,Y)-maxflow F_H in H. For each $v \in U$, note that $\deg_H(u) \leq q \cdot \deg_G(u)$. So the path decomposition of F_H routes an qA_U-respecting demand D_H. Since F_H routes D_H in H with congestion 1, D_H is routable by some flow F_G in G with congestion q. Since $\frac{1}{q}F_G$ routes $\frac{1}{q}\mathrm{mincut}_H(X,Y)$ from X to Y in G, we have $\mathrm{mincut}_G(X,Y) \geq \frac{1}{q}\mathrm{mincut}_H(X,Y)$.

2. Lecture 1, we show a flow vertex-sparsifier H for A_U of size $|E(H)| = O(|A_U| \log^2 n) = O(\deg_G(U) \log^2 n)$ and quality $4 \log n$. We are done by (1).

5 Non-trivial mincut preservers

Let $G = (V, E)$ be a *simple* graph with n vertices. Kawarabayashi and Thorup showed that there exists a graph G' such that

- G' has only $\tilde{O}(n)$ edges.
- G' is obtained by contracting several vertex sets S of G into single vertices v_S.
- G' *preserves all non-trivial mincuts* in G.

To be precise, a non-trivial mincut $(U, V \setminus U)$ is a cut such that $|U|, |V \setminus U| \geq 2$ and $(U, V \setminus U)$ is a minimum cut. The last property says that either $S \subseteq U$ or $S \cap U = \emptyset$ for all contracted sets S. This is an influential structural result that led to many exciting development in almost-linear time global minimum cut algorithms.

Simple Algorithm. The following simple algorithm gives such G' with $O(n \log n)$ edges. A better bound of $O(n)$ is known. We need two simple subroutines: For any vertex set S,

- let $\textsc{trim}(S) \subseteq S$ be obtained from S as follows: while there exists a vertex $v \in S$ where $|E(v, S)| < 2\deg(v)/5$, removes v from S;
- let $\textsc{shave}(S) = \{v \in S \mid |E(v, S)| > \deg(v)/2 + 1\}$.

The algorithm is as follows:

1. Let C be a $(\phi = 100)$-expander decomposition for $\vec{1}_V$ in G.
2. For each component X in $G - C$, let $X' = \textsc{trim}(X)$ and $X'' = \textsc{shave}(X')$.
3. Let G' be obtained from G by contracting each set X'' above.

Exercise. Let us analyze this algorithm. Let $(U, V \setminus U)$ be a non-trivial mincut. Let X be a component in $G - C$. Prove the following:

1. $\min\{|X \cap U|, |X \setminus U|\} \le \lambda/100$ where λ is the size of minimum cut of G.
2. $\min\{|X' \cap U|, |X' \setminus U|\} \le 2$.
3. $\min\{|X'' \cap U|, |X'' \setminus U|\} = 0$.
4. G' has $O(n \log(n))$ edges.
5. *Optional:* Modify the algorithm so that G' preserves all non-trivial cuts of size 1.9δ.