Sparsity, tutorial sheet

25th Max Planck Advanced Course on the Foundations of Computer Science, ADFOCS 2025

Problem 1. Consider the following two parameters of a graph G:

- *Arboricity* of G, denoted $\mathsf{arb}(G)$, is the smallest k such that the edge set of G can be partitioned into k forests.
- *Orientability* of G, denoted $\mathsf{ornt}(G)$, is the smallest ℓ such that edges of G can be oriented so that every vertex has outdegree at most ℓ.

Prove that for every graph G, we have the following:

$$\mathsf{degeneracy}(G)/2 \leqslant \mathsf{arb}(G) \leqslant \mathsf{degeneracy}(G),$$
$$\mathsf{degeneracy}(G)/2 \leqslant \mathsf{ornt}(G) \leqslant \mathsf{degeneracy}(G).$$

A *subdivision* of a graph H is a graph H' obtained by replacing every edge of H by a path. If all these paths are of length at most $\ell + 1$ (resp., exactly $\ell + 1$), then H' is an $(\leqslant \ell)$-subdivision (respectively, ℓ-subdivision) of H. We say that H is a *topological minor* of G if G contains a subdivision of H, and a *depth-d topological minor* if G contains an $(\leqslant 2d + 1)$-subdivision of G. Topological nowhere denseness and topological bounded expansion are defined exactly like the usual notions, but using topological minors.

Problem 2. Here we prove that for any graph G and $d, k \in \mathbb{N}$, if $\omega_d(G) \geqslant 2 + k^{2(d+1)}$, then $\omega^{\mathsf{top}}_{3d+1}(G) \geqslant k$.

- Suppose η is a depth-d model of K_s in G, where $s \coloneqq 2 + t^{d+1}$ and $t \coloneqq k^2$. Prove that within each branch set $\eta(u)$ one can pick a vertex $c(u)$ and a family of more than t disjoint paths of length at most $d + 1$, each leading from $c(u)$ to an edge connecting $\eta(u)$ with a different other branch set. Let $T(u)$ be the tree consisting of $c(u)$ and the union of those paths.
- Fix any k branch sets of η and within each of them construct a tree $T(u)$ as in the previous point. Show that one can use the $s - k$ other branch sets to greedily connect the highlighted trees $T(u)$ into an $(\leqslant 6d + 3)$-subdivision of K_k.

Conclude that the notions of nowhere denseness and topological nowhere denseness coincide.

Problem 3. Prove that for a graph class $\mathscr{C}$, the following conditions are equivalent:

- $\mathscr{C}$ is not nowhere dense.
- There exists $d \in \mathbb{N}$ such that for every $t \in \mathbb{N}$, some graph from $\mathscr{C}$ contains the d-subdivision of K_t as a subgraph.

Problem 4. Determine $\mathrm{wcol}_d(\mathsf{Trees})$, $\mathrm{scol}_d(\mathsf{Trees})$, and $\mathrm{adm}_d(\mathsf{Trees})$.

Problem 5. Let G be a graph, $d \in \mathbb{N}$, and $\preceq$ be a vertex ordering of G. Prove that for any two vertices u and v, the set $\mathrm{WReach}_d[u] \cap \mathrm{WReach}_d[v]$ intersects all paths between u and v of length at most d.

Problem 6. Let $\mathscr{C}$ be a class of bounded expansion and $d \in \mathbb{N}$ be fixed. Prove that given an n-vertex graph $G \in \mathbb{N}$, one can assign to every vertex u of G a label consisting of $\mathcal{O}_{\mathscr{C},d}(\log n)$ bits so that for any two vertices, from their labels one can precisely deduce their distance, provided it is at most d. Moreover, the algorithm producing the labeling should work in polynomial time.

Problem 7. Let G be a graph, $d \in \mathbb{N}$, and $\preceq$ be a vertex ordering of G with $\mathrm{wcol}_{2d}(G) \leqslant k$. For every vertex u, define the *cluster* C_u as the "inverse" weak $2d$-reachability set:

$$C_u \coloneqq \{v \mid u \in \mathrm{WReach}^{G,\preceq}_{2d}[v]\}.$$

Prove that the family $\mathcal{F} \coloneqq \{C_u \colon u \in V(G)\}$ satisfies the following properties:

- Every member of $\mathcal{F}$ is connected and has radius at most $2d$.
- For every vertex v, the radius-d ball around v is entirely contained in some member of $\mathcal{F}$.
- Every vertex of G participates in at most k members of $\mathcal{F}$.

Note: A family $\mathcal{F}$ with these properties is called a d-neighborhood cover with radius $2d$ and overlap k.

Problem 8. For a graph G, let $G \bullet c$ be the *c-blowup* of G: the graph obtained from G by replacing every vertex by a clique of size c, where cliques corresponding to adjacent vertices are joined by a complete bipartite graph. For a class $\mathscr{C}$, we denote $\mathscr{C} \bullet c := \{G \bullet c \colon G \in \mathscr{C}\}$. Prove that if $\mathscr{C}$ has bounded expansion, then so does $\mathscr{C} \bullet c$ as well, for every fixed c.

Hint: Generalized coloring numbers

Problem 9. For $c, d \in \mathbb{N}$, we say that H is a *c-congested depth-d minor* of G if there exists a *c-congested* model of H in G of depth at most d: a model where the branch sets may overlap, but every vertex belongs to at most c of them. (And branch sets of adjacent vertices should overlap or be adjacent.) For a class $\mathscr{C}$, by $\mathsf{Minors}_d^c(\mathscr{C})$ we denote the class of all c-congested depth-d minors of the graphs from $\mathscr{C}$. Prove that if $\mathscr{C}$ has bounded expansion, then so does $\mathsf{Minors}_d^c(\mathscr{C})$ as well, for any fixed $c, d \in \mathbb{N}$.

Problem 10. Given $k \in \mathbb{N}$, we say that a graph G is *k-planar* if it admits a drawing in the plane in which:
- no vertex is placed at an interior point of any edge,
- no three edges intersect at the same point, and
- each edge intersects other edges of the drawing in at most k points.

Prove that for every $k \in \mathbb{N}$, the class of k-planar graphs has bounded expansion.

Problem 11. Given a finite family $\mathcal{F}$ of (closed) balls in $\mathbb{R}^d$, we may consider its *intersection graph*: the graph on the vertex set $\mathcal{F}$ where two balls are adjacent if and only if they intersect. For $d, k \in \mathbb{N}$, let $\mathscr{B}_{d,k}$ be the class of intersection graphs of families of balls in $\mathbb{R}^d$ with *ply* at most k, that is, where every point belongs to at most k balls. Prove that for every fixed $d, k \in \mathbb{N}$, the class $\mathscr{B}_{d,k}$ has bounded expansion.

Problem 12. For a set of vertices S in a graph G, vertex $u \in S$, and $d \in \mathbb{N}$, denote by $\mathsf{backDeg}_d(u, S)$ the maximum number of paths of length at most d that start at u, end in another vertex of S, and are disjoint apart from sharing u. Consider the following algorithm computing a vertex ordering of G: starting from $S = V(G)$, iteratively extract from S a vertex with the smallest $\mathsf{backDeg}_d(u, S)$ until S becomes empty; output the reverse order of extraction. Show that the d-admissibility of this order is optimum: equal to $\mathrm{adm}_d(G)$.

Problem 13. Argue that there is a polynomial-time algorithm to d-approximate the d-admissibility of a graph.

Problem 14. Prove that for every graph G, we have $\mathsf{td}(G) = \mathrm{wcol}_\infty(G)$.

For those knowing treewidth: Prove also that $\mathsf{tw}(G) = \mathrm{scol}_\infty(G) - 1$.

Problem 15. Argue that given a p-vertex graph H and an n-vertex graph G whose treedepth is at most d, it can be decided whether G contains H as a subgraph in time $\mathcal{O}_{p,d}(n)$.

Problem 16. Let $\mathscr{C}$ be a class of bounded expansion and $d \in \mathbb{N}$ be odd. Prove that there exists a constant c, depending only on $\mathscr{C}$ and d, such that every graph from $\mathscr{C}$ admits a coloring with at most c colors so that every two vertices at distance *exactly* d receive different colors.

Hint: This is harder. First prove the statement for graphs of treedepth at most d, where the requirement is the following: any two vertices at odd distance must receive different colors. Then use low treedepth colorings.